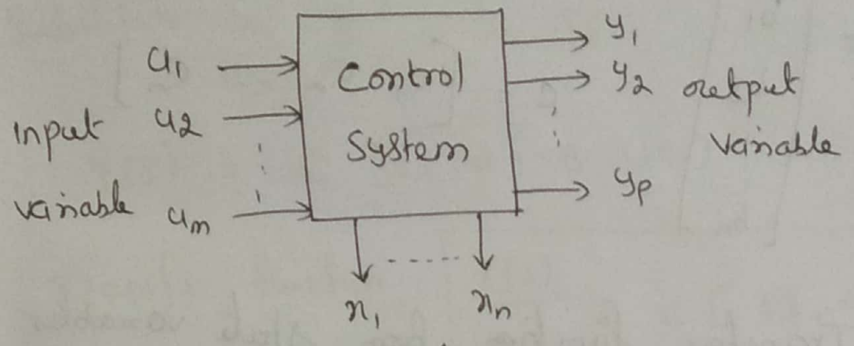
UNIT- V

Control System analysis using State Variable methods

State Space Analysis :



State model → State equations & output equations

A set of first order linear differential equation of the form

$$\dot{x}_1(t) = a_{11} x_1(t) + a_{12} x_2(t) + \cdots + a_{1n} x_n(t) + b_1 u(t)$$

$$\dot{x}_2(t) = a_{21} x_1(t) + a_{22} x_2(t) + \cdots + a_{2n} x_n(t) + b_2 u(t)$$

$$\vdots$$

$$\dot{x}_n(t) = a_{n1} x_1(t) + a_{n2} x_2(t) + \cdots + a_{nn} x_n(t) + b_n u(t)$$

The output of the system can be expressed as a linear combination of state variables

$$y(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots c_n x_n(t)$$

$$\therefore \boxed{\begin{aligned} \dot{x}(t) &= A x(t) + B u(t) \\ Y(t) &= C X(t) + D u(t) \end{aligned}}$$

where,

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

* Transfer function from state variables

Let

$$\dot{x}(t) = A\,x(t) + B\,u(t) \rightarrow (1)$$

$$y(t) = C\,x(t) + D\,u(t)$$

put $D=0$,

$$\therefore y(t) = C\,x(t) \rightarrow (2)$$

Taking Laplace transform of eq's (1) & (2)

$$S\,X(S) = A\,X(S) + B\,U(S) \rightarrow (3)$$

$$Y(S) = C\,X(S) \rightarrow (4)$$

$$\therefore S\,X(S) - A\,X(S) = B\,U(S)$$

$$X(S)\,[S - A] = B\,U(S)$$

multiply 'S' with identity matrix

$$X(s)[sI - A] = B\,U(s)$$

$$\therefore X(s) = \frac{B\,U(s)}{(sI-A)}$$

$$X(s) = [sI - A]^{-1} B\,U(s) \rightarrow (5)$$

Substitute eq (5) in to (4).

$$Y(s) = C\,[sI - A]^{-1} B\,U(s)$$

$\therefore$ Transfer function, $\boxed{\dfrac{Y(s)}{U(s)} = C\,[sI - A]^{-1} B}$

$$\therefore \boxed{\frac{Y(s)}{U(s)} = C\,\frac{\operatorname{Adj}(sI-A)}{|sI-A|}\,B}$$

Ex 1 Obtain the transfer function of the system defined by following state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:-

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$y(t) = c(t)$$

$$T.F, \quad \frac{Y(s)}{U(s)} = C\,[sI - A]^{-1} B$$

To find $[SI-A]^{-1}$:

$$[SI-A] = S\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}-2 & 1\\1 & -2\end{bmatrix}$$

$$= \begin{bmatrix}S & 0\\0 & S\end{bmatrix} - \begin{bmatrix}-2 & 1\\1 & -2\end{bmatrix} = \begin{bmatrix}S+2 & -1\\-1 & S+2\end{bmatrix}$$

$$\therefore [SI-A]^{-1} = \frac{\mathrm{Adj}\,(SI-A)}{|SI-A|}$$

$$|SI-A| = \begin{vmatrix}S+2 & -1\\-1 & S+2\end{vmatrix} = (S+2)(S+2) - 1$$

$$= S^2 + 4S + 4 - 1$$

$$= S^2 + 4S + 3$$

$$\mathrm{Adj}\,(SI-A) = \begin{bmatrix}S+2 & -1\\-1 & S+2\end{bmatrix}$$

* Interchange a_{11} & a_{22}
Change sign of a_{12} & a_{21}

$$= \begin{bmatrix}S+2 & 1\\1 & S+2\end{bmatrix}$$

$$\therefore [SI-A]^{-1} = \frac{\begin{bmatrix}S+2 & 1\\1 & S+2\end{bmatrix}}{S^2+4S+3}$$

$\therefore$ To find Transfer function

$$\frac{Y(S)}{U(S)} = C\,[SI-A]^{-1}\,B$$

$$= \frac{[0 \;\; 1]}{S^2+4S+3}\begin{bmatrix}S+2 & 1\\1 & S+2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{[0 \quad 1]}{s^2+4s+3} \begin{bmatrix} s+2 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{\text{T.F}, \quad \frac{Y(s)}{U(s)} = \frac{1}{s^2+4s+3}}$$

Ex 2 :- Obtain the transfer function of the following state equations.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [0 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution :

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$y(t) = C\,x(t)$$

$$\text{T.F}, \quad \frac{Y(s)}{U(s)} = C\,[sI-A]^{-1} B$$

To find $[sI-A]^{-1}$:-

$$[sI-A] = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} s+2 & -1 & 0 \\ 0 & s+3 & -1 \\ 3 & 4 & s+5 \end{bmatrix}$$

$$\therefore [SI - A]^{-1} = \frac{Adj\,(SI-A)}{|SI-A|}$$

$$|SI-A| = \begin{vmatrix} S+2 & -1 & 0 \\ 0 & S+3 & -1 \\ 3 & 4 & S+5 \end{vmatrix}$$

$$= (S+2)\left[(S+3)(S+5)+4\right] + (1)(3) + 0$$

$$= (S+2)(s^2+8s+19) + 3$$

$$= s^3 + 8s^2 + 19s + 2s^2 + 16s + 38 + 3$$

$$= s^3 + 10s^2 + 35s + 41$$

<u>To find</u> <u>Adj (SI-A):</u>

$$\text{co-}\cancel{\text{factor}}\text{ of } (SI-A) = \begin{bmatrix} +\begin{vmatrix} S+3 & -1 \\ 4 & S+5 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ 3 & S+5 \end{vmatrix} & +\begin{vmatrix} 0 & S+3 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} -1 & 0 \\ 4 & S+5 \end{vmatrix} & +\begin{vmatrix} S+2 & 0 \\ 3 & S+5 \end{vmatrix} & -\begin{vmatrix} S+2 & -1 \\ 3 & 4 \end{vmatrix} \\ +\begin{vmatrix} -1 & 0 \\ S+3 & -1 \end{vmatrix} & -\begin{vmatrix} S+2 & 0 \\ 0 & -1 \end{vmatrix} & +\begin{vmatrix} S+2 & -1 \\ 0 & S+3 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (S+3)(S+5)+4 & -3 & -3(S+3) \\ S+5 & (S+2)(S+5) & -[4(S+2)+3] \\ 1 & S+2 & (S+2)(S+3) \end{bmatrix}$$

$$= \begin{bmatrix} s^2+8s+19 & -3 & -3s-9 \\ S+5 & s^2+7s+10 & -4s-11 \\ 1 & S+2 & s^2+5s+6 \end{bmatrix}$$

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$$\text{Adj}[SI-A] = [\text{co-factor matrix}]^T$$

$$= \begin{bmatrix} s^2+8s+19 & s+5 & 1 \\ -3 & s^2+7s+10 & s+2 \\ -3s-19 & -4s-11 & s^2+5s+6 \end{bmatrix}$$

$$\therefore [SI-A]^{-1} = \frac{\text{Adj}[SI-A]}{|SI-A|}$$

$$= \frac{1}{s^3+10s^2+35s+41} \begin{bmatrix} s^2+8s+19 & s+5 & 1 \\ -3 & s^2+7s+10 & s+2 \\ -3s-19 & -4s-11 & s^2+5s+6 \end{bmatrix}$$

To find T.F :-

$$\text{T. F}, \quad \frac{Y(s)}{U(s)} = C\,[SI-A]^{-1}\,B$$

$$\frac{Y(s)}{U(s)} = \frac{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}{s^3+10s^2+35s+41} \begin{bmatrix} s^2+8s+19 & s+5 & 1 \\ -3 & s^2+7s+10 & s+2 \\ -3s-19 & -4s-11 & s^2+5s+6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}{s^3+10s^2+35s+41} \begin{bmatrix} 1 \\ s+2 \\ s^2+5s+6 \end{bmatrix}$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{s+2}{s^3+10s^2+35s+41}}$$

* State space representation of Electrical network

Procedure :-

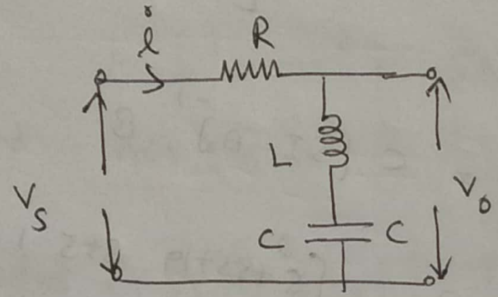
i) Applying KVL,

ii) Choose I_L and V_C as state variable

iii) choose V_R as output, y

iv) choose the existing sources V_s as input, u

Ex 1 Obtain the state space representation of electrical network shown in figure.



Solution :-

Step 1 Applying KVL,

$$V_s = R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt \rightarrow (1)$$

$$V_c = \frac{1}{C}\int i(t)\,dt \rightarrow (2)$$

Substitute eq (2) in to eq (1),

$$V_s = R\,i(t) + L\frac{di(t)}{dt} + V_c$$

$$L\frac{di(t)}{dt} = V_s - R\,i(t) - V_c$$

$$\frac{di(t)}{dt} = \frac{V_s}{L} - \frac{R}{L}i(t) - \frac{1}{L}V_c \rightarrow (3)$$

Step 2 : Choose current through inductor & voltage across capacitor as state variable.

$i = x_1 \rightarrow \frac{di}{dt} = \dot{x}_1$ and $V_s = u$.

$V_c = x_2$

Substitute above state variable in eq ② & eq ③.

$$x_2 = \frac{1}{C}\int x_1 \, dt$$

$$\dot{x}_2 = \frac{1}{C} x_1$$

$$\dot{x}_1 = \frac{1}{L} u - \frac{R}{L} x_1 - \frac{1}{L} x_2$$

State equations :-

$$\boxed{\begin{aligned} \dot{x}_1 &= -\frac{R}{L} x_1 - \frac{1}{L} x_2 + \frac{1}{L} u \\ \dot{x}_2 &= \frac{1}{C} x_1 \end{aligned}}$$

In matrix form,

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u}$$

Step 3 : choose voltage across resistor as output

$$y = i(t) R$$

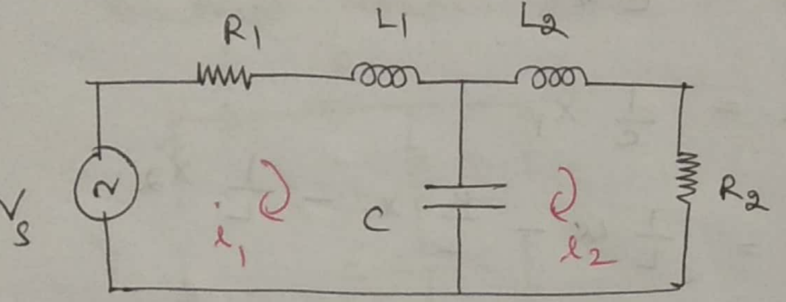
$$y = x_1 R$$

$$\boxed{Y = R\,X_1}$$

In matrix form,

$$\boxed{[Y] = [R \quad 0]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}$$

Ex Q: Determine the state space representation of the electrical network shown in figure.



Solution:-

Step 1: Applying KVL,

$$V_s = R_1 i_1 + L_1 \frac{di_1}{dt} + \frac{1}{C}\int (i_1 - i_2)\,dt \quad \to (1)$$

$$0 = R_2 i_2 + L_2 \frac{di_2}{dt} + \frac{1}{C}\int (i_2 - i_1)\,dt \quad \to (2)$$

and

$$V_c = \frac{1}{C}\int (i_1 - i_2)\,dt \quad \to (3)$$

~~Step 2~~: Substitute eq (3) in to eq's (1) & (2),

$$V_s = R_1 i_1 + L_1 \frac{di_1}{dt} + V_c \quad \to (4)$$

$$0 = R_2 i_2 + L_2 \frac{di_2}{dt} + V_c \quad \to (5)$$

Step-2 :- Choose current through inductor and voltage across capacitor as state variable.

$i_1 = x_1 \quad \rightarrow \quad \frac{di_1}{dt} = \dot{x}_1 \qquad$ and $\quad V_s = u$

$i_2 = x_2 \quad \rightarrow \quad \frac{di_2}{dt} = \dot{x}_2$

$v_c = x_3$

Substitute above state variables in eq's (4) & (5) and (3)

$$u = R_1 x_1 + L_1 \dot{x}_1 + x_3 \rightarrow (6)$$

$$0 = R_2 x_2 + L_2 \dot{x}_2 + x_3 \rightarrow (7)$$

$$x_3 = \frac{1}{c} \int (x_1 - x_2)\, dt \rightarrow (8)$$

Differentiate eq (8) w.r.t. 't'

$$\dot{x}_3 = \frac{1}{c} x_1 - \frac{1}{c} x_2$$

State equations :-

$$L_1 \dot{x}_1 = u - R_1 x_1 - x_3$$

$$\boxed{\dot{x}_1 = -\frac{R_1}{L} x_1 - \frac{1}{L_1} x_3 + \frac{1}{L} u}$$

$$L_2 \dot{x}_2 = -R_2 x_2 - x_3$$

$$\boxed{\dot{x}_2 = -\frac{R_2}{L_2} x_2 - \frac{1}{L_2} x_3}$$

$$\boxed{\dot{x}_3 = \frac{1}{c} x_1 - \frac{1}{c} x_2}$$

In matrix form,

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & \frac{-1}{L_1} \\ 0 & \frac{-R_2}{L_2} & \frac{-1}{L_2} \\ 1/c & -1/c & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/L_1 \\ 0 \\ 0 \end{bmatrix} u}$$

Step-3 :- Choose voltage across resistors as output variable.

$y_1 = i_1 R_1$

$y_2 = i_2 R_2$

$$\boxed{\begin{aligned} y_1 &= R_1 x_1 \\ y_2 &= R_2 x_2 \end{aligned}}$$

$$\boxed{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

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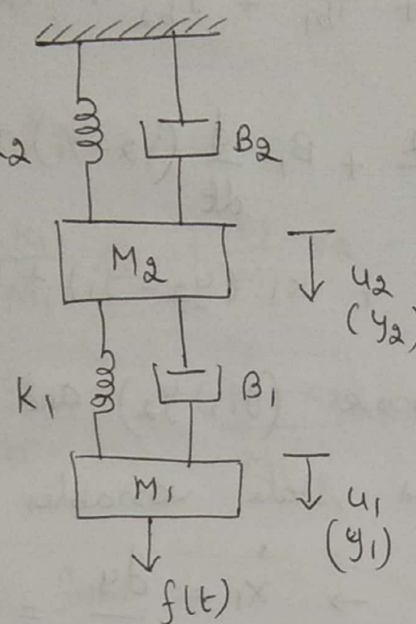
* State Space Representation of Mechanical system :-

i) Applying Newton's second law

ii) Choose x, u or a as X.

iii) Choose 'x' as output & 'f' as input.

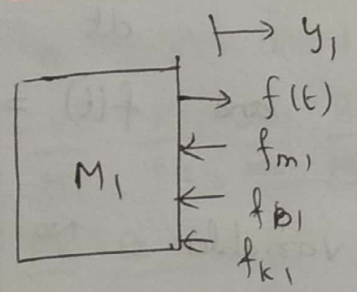
Ex 1:- obtain the State Space representation of the following mechanical system.



[Note: if displacement (y) not given, then you need to assign]

Solution :-

Step 1 :- Free body diagram of M_1



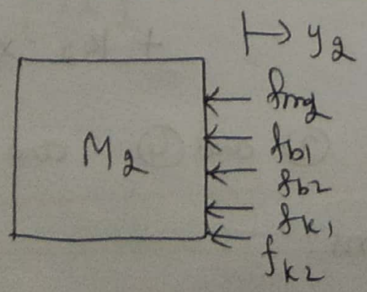
$u_1 \rightarrow$ velocity
$y_1 \rightarrow$ displacement

Applying Newton's second law,

$$f(t) = f_{m1} + f_{b1} + f_{k1}$$

$$\therefore f(t) = M_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{d}{dt}(y_1 - y_2) + k_1 (y_1 - y_2) \quad \text{(1)}$$

Step 2 :- Free body diagram of M_2



Applying Newton's second law,

$$0 = f_{m2} + f_{b1} + f_{b2} + f_{k1} + f_{k2}$$

$$0 = M_2 \frac{d^2 y_2}{dt} + B_1 \frac{d}{dt}(y_2 - y_1) + B_2 \frac{dy_2}{dt}$$

$$+ k_1 (y_2 - y_1) + k_2 y_2 \rightarrow (2)$$

Step 3:- choose (y_1, y_2) and $\left(\frac{dy_1}{dt}, \frac{dy_2}{dt}\right)$ as state variables

$x_1 = y_1 \rightarrow \dot{x}_1 = \frac{dy_1}{dt} = x_3$; $\dot{x}_3 = \frac{d^2 y_1}{dt^2}$

$x_2 = y_2 \rightarrow \dot{x}_2 = \frac{dy_2}{dt} = x_4$; $\dot{x}_4 = \frac{d^2 y_2}{dt^2}$

$x_3 = \frac{dy_1}{dt}$

$x_4 = \frac{dy_2}{dt}$ and $f(t) = u$

Note : No. of variables = No. of displacements + No. of velocities

$= 2 + 2 = 4$

Substitute above state variables in eq's (1) & (2)

$$u = M_1 \dot{x}_3 + B_1 (x_3 - x_4) + k_1 (x_1 - x_2) \rightarrow (3)$$

$$0 = M_2 \dot{x}_4 + B_1 (x_4 - x_3) + B_2 x_4 + k_1 (x_2 - x_1)$$

$$+ k_2 x_2 \rightarrow (4)$$

Rearranging eq's (3) and (4) and write the state equations.

State equations :-

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = -\frac{k_1}{M_1}x_1 + \frac{k_1}{M_1}x_2 - \frac{B_1}{M_1}x_3 + \frac{B_1}{M_1}x_4 + \frac{1}{M_1}u$$

$$\dot{x}_4 = \frac{k_1}{M_2}x_1 - \frac{(k_1+k_2)}{M_2}x_2 + \frac{B_1}{M_2}x_3 - \frac{(B_1+B_2)}{M_2}x_4$$

In matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{M_1} & \frac{K_1}{M_1} & -\frac{B_1}{M_1} & \frac{B_1}{M_1} \\ \frac{k_1}{M_2} & -\frac{(k_1+k_2)}{M_2} & \frac{B_1}{M_2} & -\frac{(B_1+B_2)}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix} u$$

Step 4:

Displacements are taken as output

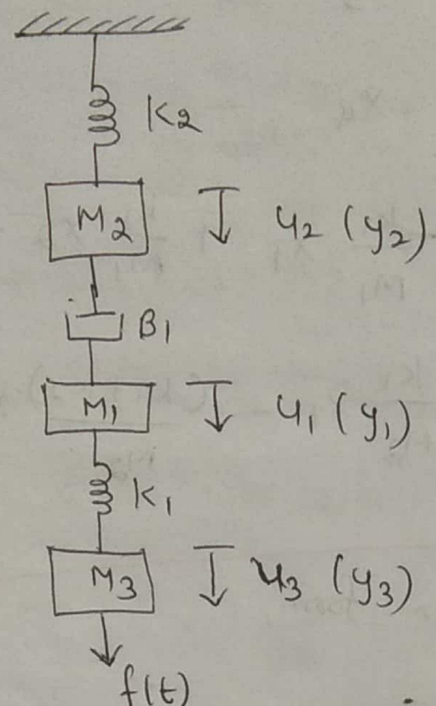
$$y_1 = x_1$$

$$y_2 = x_2$$

In matrix form,

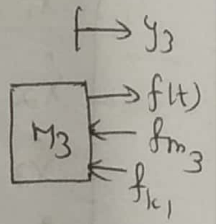
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Ex 2 Determine the signal space representation of the following mechanical system.



Solution

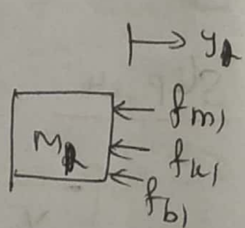
Step 1: Free body diagram of M_3



$$f(t) = f_{m_3} + f_{k_1}$$

$$f(t) = M_3 \frac{d^2 y_3}{dt^2} + k_1 (y_3 - y_1) \rightarrow ①$$

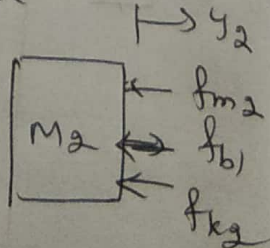
Step 2: Free body diagram of M_1



$$0 = f_{m_1} + f_{b_1} + f_{k_1}$$

$$0 = M_1 \frac{d^2 y_1}{dt^2} + B_1 \frac{d}{dt}(y_1 - y_2) + k_1 (y_1 - y_3) \rightarrow ②$$

Step 3: Free body diagram of M_2



$$0 = f_{m_2} + f_{b_1} + f_{k_2}$$

$$0 = M_2 \frac{d^2 y_2}{dt^2} + B_1 \frac{d}{dt}(y_2 - y_1) + k_2 y_2 \rightarrow ③$$

Step 4 choose the (y_1, y_2, y_3) & $\left(\frac{dy_1}{dt}, \frac{dy_2}{dt}, \frac{dy_3}{dt}\right)$ as state variable.

No. of state variables = No. of displacements + No. of velocities

$= 3 + 3 = 6$

$x_1 = y_1 \Rightarrow \dot{x}_1 = \frac{dy_1}{dt} = x_4$ and $f(t) = u$.

$x_2 = y_2 \Rightarrow \dot{x}_2 = \frac{dy_2}{dt} = x_5$

$x_3 = y_3 \Rightarrow \dot{x}_3 = \frac{dy_3}{dt} = x_6$

$x_4 = \frac{dy_1}{dt} \Rightarrow \dot{x}_4 = \frac{d^2y_1}{dt^2}$

$x_5 = dy_2/dt \Rightarrow \dot{x}_5 = d^2y_2/dt$

$x_6 = dy_3/dt \Rightarrow \dot{x}_6 = \frac{d^2y_3}{dt^2}$

Substitute above state variables in eq's ①, ② & ③

$$u = M_3 \dot{x}_6 + k_1 (x_3 - x_1)$$

$$0 = M_1 \dot{x}_4 + B_1 (x_4 - x_5) + k_1 (x_1 - x_3)$$

$$0 = M_2 \dot{x}_5 + B_1 (x_5 - x_4) + k_2 x_2$$

State equations:

$$\dot{x}_1 = x_4$$

$$\dot{x}_2 = x_5$$

$$\dot{x}_3 = x_6$$

$$\dot{x}_4 = \frac{-k_1}{M_1} x_1 + \frac{k_1}{M_1} x_3 - \frac{B_1}{M_1} x_4 + \frac{B_1}{M_1} x_5$$

$$\dot{X}_5 = -\frac{k_2}{M_2} x_2 + \frac{B_1}{M_2} x_4 - \frac{B_1}{M_2} x_5$$

$$\dot{X}_6 = -\frac{k_1}{M_3} x_1 + \frac{k_1}{M_3} x_3 + \frac{1}{M_3} u.$$

In matrix form,

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k_1}{M_1} & 0 & \frac{k_1}{M_1} & -\frac{B_1}{M_1} & \frac{B_1}{M_1} & 0 \\ 0 & -\frac{k_2}{M_2} & 0 & \frac{B_1}{M_2} & -\frac{B_1}{M_2} & 0 \\ -\frac{k_1}{M_3} & 0 & \frac{k_1}{M_3} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{M_3} \end{bmatrix} u}$$

Step 5 :- Choose displacements as output

$y_1 = x_1$

$y_2 = x_2$

$y_3 = x_3$

In matrix form

$$\boxed{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

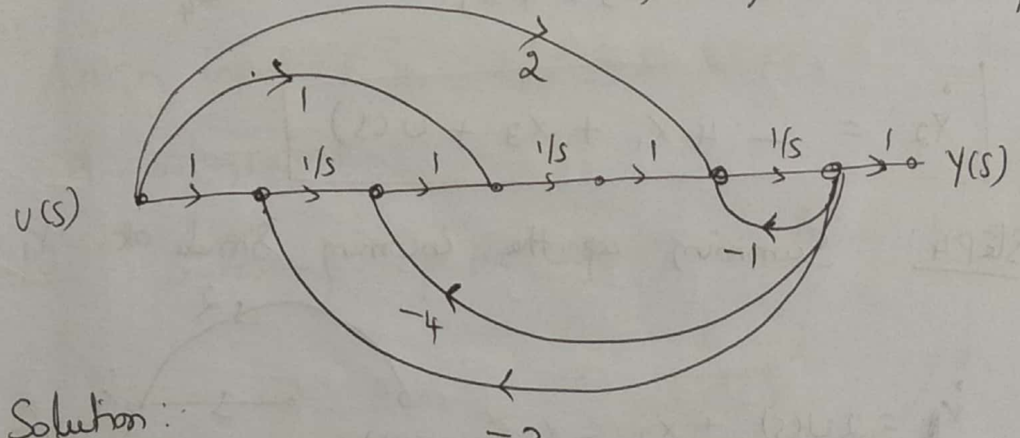
* Signal Space representation of Signal Flow Graph :-

i) Summing up the incoming signal at each input node of integrator.

ii) Summing up the incoming signals at output node

iii) Write the state and output equations in matrix form.

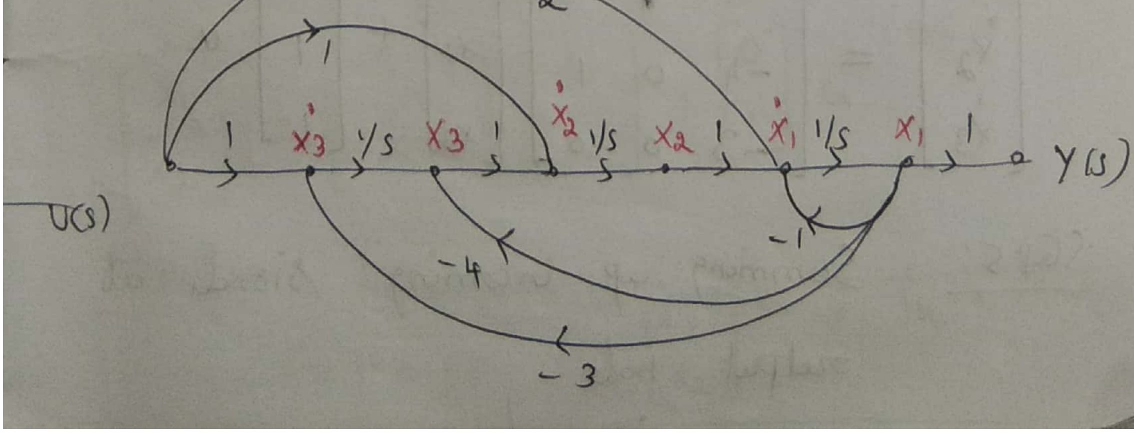
Ex 1 :- Construct the state space representation from the following signal flow graph.



Solution :

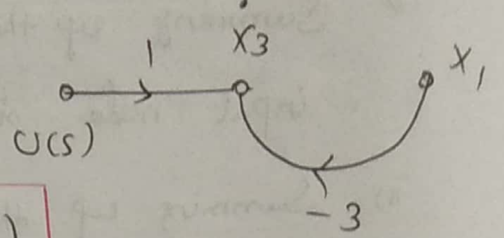
No. of state variables = No. of integrators

Step 1 :- Assign input and output of integrators as $(\dot{x}_1, \dot{x}_2, \dot{x}_3)$ and (x_1, x_2, x_3)



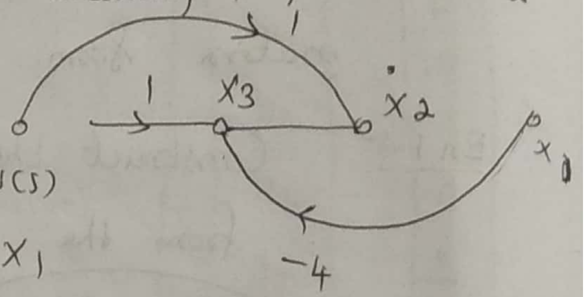
Step 2: Summing up the incoming signal at input node $\dot{x}_3$ of integrator.

$$\dot{x}_3 = U(s) - 3x_1$$



$$\therefore \boxed{\dot{x}_3 = -3x_1 + U(s)}$$

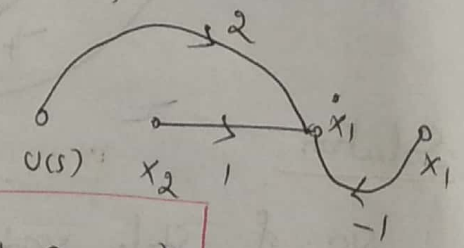
Step 3: Summing up the incoming signals at $\dot{x}_2$



$$\dot{x}_2 = U(s) + x_3 - 4x_1$$

$$\boxed{\dot{x}_2 = -4x_1 + x_3 + U(s)}$$

Step 4: Summing up the incoming signals at $\dot{x}_1$



$$\dot{x}_1 = 2U(s) + x_2 - x_1$$

$$\boxed{\dot{x}_1 = -x_1 + x_2 + 2U(s)}$$

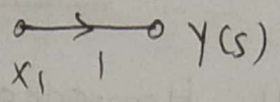
In matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -4 & 0 & 1 \\ -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u$$

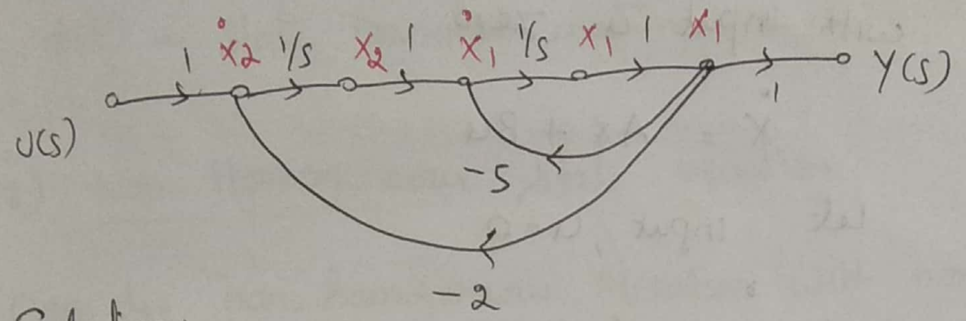
Step 5:- Summing up incoming signals at output node

$Y = x_1$

$$[Y] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Ex 2:



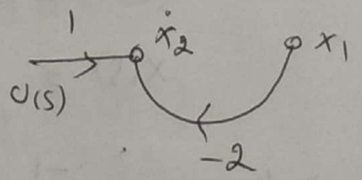
Solution:-

No. of state variables = No. of integrators
$= 2$

Step 1:- Assign input $(\dot{x}_1, \dot{x}_2)$ and output (x_1, x_2) of integrators.

Step 2:- Summing up the incoming signals of $\dot{x}_2$

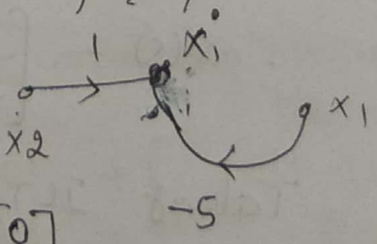
$\dot{x}_2 = U(s) - 2x_1$



$$\boxed{\therefore \dot{x}_2 = -2x_1 + U(s)}$$

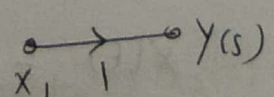
Step 3:- Summing up the incoming signals at $\dot{x}_1$

$$\boxed{\dot{x}_1 = -5x_1 + x_2}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Step 4:- $\boxed{Y = x_1}$



$$[Y] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

* Solutions of State equations

(i) Homogeneous state equations :-

Consider the homogeneous state equation with input is zero.

$$\dot{X} = AX + Bu$$

Let input, $u = 0$

$$\therefore \dot{X} = AX$$

Taking LT, with initial conditions,

$$S\,X(s) - \cancel{X(0)} = A\,X(s)$$

$$S\,X(s) - A\,X(s) = X(0)$$

$$X(S)\,[SI - A] = X(0)$$

$$X(S) = [SI - A]^{-1} X(0)$$

But $$[SI - A]^{-1} = \left[\frac{1}{S} + \frac{A}{S^2} + \frac{A}{S^3} + \cdots\right] = \phi(s)$$

$$\therefore X(S) = \left[\frac{1}{S} + \frac{A}{S^2} + \frac{A}{S^3} + \cdots\right] X(0)$$

Taking ILT,

$$X(t) = \left[1 + At + A\frac{t^2}{2!} + \cdots\right] X(0)$$

$$X(t) = e^{At}\,X(0)$$

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where, $e^{At} = \left[1 + At + A\frac{t^2}{2!} + \cdots\right] = L^{-1}[SI - A]^{-1}$

$\therefore L^{-1}[\phi(s)] = \phi(t)$

$$\therefore \boxed{X(t) = \phi(t)\, X(0)}$$

$\phi(t)$ = State transition matrix $= L^{-1}[SI - A]^{-1}$

(ii) Non-Homogeneous State equation:

Consider non-homogeneous equation with non-zero input

$$\dot{X} = AX + Bu$$

Taking LT with initial conditions,

$$S\,X(S) - X(0) = A\,X(S) + B\,U(S)$$

$$S\,X(S) - A\,X(S) = X(0) + B\,U(S)$$

$$(SI - A)\,X(S) = X(0) + B\,U(S)$$

$$X(S) = (SI - A)^{-1} X(0) + (SI - A)^{-1} B\,U(S)$$

~~But~~ Taking ILT using convolution integral theorem,

$$X(t) = L^{-1}[SI - A]^{-1} X(0) + \int_0^t e^{A(t-T)} B\,U(T)\,dT$$

$$X(t) = e^{At} X(0) + \int_0^t e^{A(t-T)} B\,U(T)\,dT$$

Note: $L^{-1}[x_1(t) \times x_2(t)] = \int_0^t x_1(t-T)\, x_2(T)\, dT$.

Ex1:- Obtain the signal space representation of the system $y'' + 3y' + 2y = 0$

Find $\phi(t)$ & $x(t)$ given $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution:-

Choose state variables as y and y'

$$x_1 = y$$

$$x_2 = y'$$

Differentiate above state variables,

$$x_1' = y' = x_2$$

$$x_2' = y'' = -3y' - 2y$$

$$x_2' = -2x_1 - 3x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

output equation : $y = x_1$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To find $\phi(t)$:-

$$\phi(t) = e^{At} = L^{-1}[sI - A]^{-1}$$

$$[SI - A] = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$

$$[SI - A]^{-1} = \frac{Adj(SI - A)}{|SI - A|}$$

$$|SI - A| = \begin{vmatrix} S & -1 \\ 2 & S+3 \end{vmatrix} = S(S+3) + 2$$
$$= S^2 + 3S + 2 = (S+1)(S+2)$$

$$Adj(SI - A) = \begin{vmatrix} S & -1 \\ 2 & S+3 \end{vmatrix} = \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$

$$\therefore [SI - A]^{-1} = \begin{bmatrix} \frac{S+3}{(S+1)(S+2)} & \frac{1}{(S+1)(S+2)} \\ \frac{-2}{(S+1)(S+2)} & \frac{S}{(S+1)(S+2)} \end{bmatrix}$$

To find $\phi(t)$, apply PFE.

$$(SI - A)^{-1} = \begin{bmatrix} \frac{A_1}{S+1} + \frac{B_1}{S+2} & \frac{A_2}{S+1} + \frac{B_2}{S+2} \\ \frac{A_3}{S+1} + \frac{B_3}{S+2} & \frac{A_4}{S+1} + \frac{B_4}{S+2} \end{bmatrix}$$

$$\frac{S+3}{(S+1)(S+2)} = \frac{A_1}{S+1} + \frac{B_1}{S+2}$$

$$S+3 = A_1(S+2) + B_1(S+1)$$

Put $S = -1$, $\boxed{A_1 = 2}$

Put $S = -2$, $\boxed{B_1 = -1}$

$$\frac{1}{(S+1)(S+2)} = \frac{A_2}{S+1} + \frac{B_2}{S+2}$$

$$1 = A_2(S+2) + B_2(S+1)$$

Put $S = -2$, $\boxed{B_2 = -1}$

$S = -1$ $\boxed{A_2 = +1}$

$$\frac{-2}{(S+1)(S+2)} = \frac{A_3}{S+1} + \frac{B_3}{S+2}$$

$$-2 = A_3(S+2) + B_3(S+1)$$

put

$S=-1$, $\boxed{A_3 = -2}$

$S=-2$, $\boxed{B_3 = 2}$

$$\frac{S}{(S+1)(S+2)} = \frac{A_4}{S+1} + \frac{B_4}{S+2}$$

$$S = A_4(S+2) + B_4(S+1)$$

Put

$S=-1$ $A_4 = -1$

$S=-2$ $B_4 = 2$

$$\therefore \phi(t) = L^{-1}\left[SI - A\right]^{-1}$$

$$= L^{-1}\begin{bmatrix} \frac{2}{S+1} - \frac{1}{S+2} & \frac{1}{S+1} - \frac{1}{S+2} \\ \frac{-2}{S+1} + \frac{2}{S+2} & \frac{-1}{S+1} + \frac{2}{S+2} \end{bmatrix}$$

$$\boxed{\phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}}$$

To find $X(t)$:-

$$X(t) = \phi(t) \cdot X(0)$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{\therefore X(t) = \begin{bmatrix} 2e^{-t} - 2e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix}}$$

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Ex : 2 Find $x(t)$ for the given State Space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where $u(t)$ is unit step input. The initial conditions are $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

Solution :-

$$\dot{x} = Ax + Bu$$

For non-homogeneous state equation,

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

To find e^{At} :-

$$e^{At} = L^{-1} [sI - A]^{-1}$$

$$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{\text{Adj}\,(sI - A)}{|sI - A|}$$

$$|sI - A| = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s(s+3) + 2$$

$$= s^2 + 3s + 2$$

$$= (s+1)(s+2)$$

$$Adj\,(SI - A) = \begin{bmatrix} S+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\therefore [SI - A]^{-1} = \begin{bmatrix} \dfrac{S+3}{(S+1)(S+2)} & \dfrac{1}{(S+1)(S+2)} \\[2ex] \dfrac{-2}{(S+1)(S+2)} & \dfrac{S}{(S+1)(S+2)} \end{bmatrix}$$

Applying partial fraction expansion,

$$[SI - A]^{-1} = \begin{bmatrix} \dfrac{A_1}{S+1} + \dfrac{B_1}{S+2} & \dfrac{A_2}{S+1} + \dfrac{B_2}{S+2} \\[2ex] \dfrac{A_3}{S+1} + \dfrac{B_3}{S+2} & \dfrac{A_4}{S+1} + \dfrac{B_4}{S+2} \end{bmatrix}$$

$S+3 = A_1(S+2) + B_2(S+1)$

$$\boxed{\begin{array}{l} B_1 = -1 \\ A_1 = 2 \end{array}}$$

$1 = A_2(S+2) + B_2(S+1)$

$$\boxed{\begin{array}{l} A_2 = 1 \\ B_2 = -1 \end{array}}$$

$-2 = A_3(S+2) + B_3(S+1)$

$$\boxed{A_3 = -2, \quad B_3 = 2}$$

$S = A_4\,(S+2) + B_4(S+1)$

$$\boxed{A_4 = -1 \qquad B_4 = +2}$$

$$\therefore [SI - A]^{-1} = \begin{bmatrix} \dfrac{2}{S+1} - \dfrac{1}{S+2} & \dfrac{1}{S+1} - \dfrac{1}{S+2} \\[2ex] \dfrac{-2}{S+1} + \dfrac{2}{S+2} & \dfrac{-1}{S+1} + \dfrac{2}{S+2} \end{bmatrix}$$

$$\phi(t) = e^{At} = L^{-1}[SI - A]^{-1}$$

Taking ILT,

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\therefore e^{At} x(0) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau = \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} e^{-t} . e^{\tau} - e^{-2t} . e^{2\tau} \\ -e^{-t} . e^{\tau} + 2e^{-2t} e^{2\tau} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} e^{-t} e^{\tau} - e^{-2t} \dfrac{e^{2\tau}}{2} \\ -e^{-t} e^{\tau} + \cancel{2} e^{-2t} \dfrac{e^{2\tau}}{\cancel{2}} \end{bmatrix}_0^t$$

$$= \begin{bmatrix} e^{-t} . e^{t} - e^{-2t} . \dfrac{e^{2t}}{2} - e^{-t} + \dfrac{e^{-2t}}{2} \\ -e^{-t} . e^{t} + e^{-2t} . e^{2t} + e^{-t} - e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \\ -1 + 1 + e^{-t} - e^{-2t} \end{bmatrix}$$

$$\int_0^t e^{A(t-\tau)} B u(\tau) d\tau = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$\therefore \quad x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$\boxed{x(t) = \begin{bmatrix} \frac{1}{2} + e^{-t} - \frac{1}{2} e^{-2t} \\ -e^{-t} + e^{-2t} \end{bmatrix}}$$

Concepts of controllability & observability

Kalman's test

For controllability,

$$Q_c = [B \quad AB \quad A^2B + \cdots \quad A^{n-1}B]$$

For observability

$$Q_o = [C^T \quad A^T C^T \quad (A^T)^2 C^T \cdots \quad (A^T)^{n-1} C^T]$$

Ex 1 :- Find the controllability of the System described by the State equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Solution :-

$$\dot{x} = AX + Bu \quad ; \quad n = 2$$

$$Q_c = \begin{bmatrix} B & AB & A^2B & --- & A^{n-1}B \end{bmatrix}$$

put $n = 2$

$$\therefore Q_c = \begin{bmatrix} B & AB \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\therefore |Q_c| = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1 \neq 0 \Rightarrow \text{non-singular matrix}$$

So the given system is completely State controllable.

Ex 2 :- Determine the State controllability and observability of the system described by

$$Y = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x$$

$$\dot{x} = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} u$$

Solution :-

$$\dot{x} = Ax + Bu \quad ; \quad n = 3$$

$$y = Cx$$

To find controllability :

$$Q_c = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

$$Q_c = [B \quad AB \quad A^2B]$$

$$A^2B = A(AB).$$

$$AB = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A^2B = A(AB) = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 0 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\therefore Q_c = \begin{bmatrix} 0 & 1 & 2 & -2 & -2 & 7 \\ 0 & 0 & 2 & 0 & 0 & 3 \\ 2 & 1 & 2 & 1 & 2 & 1 \end{bmatrix}$$

Take any 3×3 matrix & find its rank.

$$Q_{c,sub} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 2 & 1 & 2 \end{bmatrix}$$

$$|Q_{c,sub}| = \begin{vmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 2 & 1 & 2 \end{vmatrix} = 0 - 1(0-4) + 2(0)$$
$$= 4 \neq 0$$

$\therefore$ Rank of $Q_c = 3$ $\Rightarrow$ non singular matrix

Hence, the system is completely state controllable.

To find observability :

$$Q_o = \left[C^T \quad A^T C^T \quad (A^T)^2 C^T \ldots (A^T)^{n-1} C^T \right]$$

Put $n = 3$

$$Q_o = \left[C^T \quad A^T C^T \quad (A^T)^2 C^T \right]$$

$$[A^T C^T] = \begin{bmatrix} -3 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^T)^2 C^T = A^T (A^T C^T) = \begin{bmatrix} -3 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -4 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 11 \\ 0 & -4 \\ 1 & -1 \end{bmatrix}$$

$$\therefore \quad Q_o = \begin{bmatrix} 0 & 1 & 0 & -4 & 0 & 11 \\ 0 & 1 & 0 & 1 & 0 & -4 \\ 1 & 0 & 1 & 2 & 1 & -1 \end{bmatrix}$$

Take any 3x3 matrix & find its rank

$$|Q_{o,sub}| = \begin{vmatrix} 1 & 0 & -4 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1(0-1) - 0 - 4(1)$$
$$= -1 - 4 = -5 \neq 0$$

Rank of $Q_c = 3$ $\Rightarrow$ non singular matrix

$\therefore$ Hence the system is completely state observable.

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Ex: 3 Obtain the State Space model for an LTI system whose transfer function is given by

$$G(s) = \frac{-2s+1}{s^3+5s^2+3s+1} = \frac{Y(s)}{U(s)}$$

Solution:-

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$= \frac{-2s+1}{s^3+5s^2+3s+1}$$

on comparing, we get

$b_2 = -2$; $b_3 = 1$

$a_1 = 5$; $a_2 = 3$; $a_3 = 1$

State equation in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_3 - a_3 b_0 \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} u$$

To find controllability:-

$$Q_c = \begin{bmatrix} B & AB & A^2 B \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$A^2B = A(AB) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 11 \end{bmatrix}$$

$$\therefore Q_c = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 6 \\ 0 & -2 & 11 \end{bmatrix}$$

$$|Q_c| = 1(11+12) - 0 + 2(4) = 31 \neq 0$$

$\Rightarrow$ Non-singular matrix

Hence, the given system is completely state controllable.