Unit-5.

Lattices and Boolean Algebra

partial Order Relation:

Let 'X' be any set, 'R' be a relation defined on 'X'. The 'R' is said to be partial order Relation. If it satisfies reflective, antisymmetric, Transitive relations.

i) $xRx \Rightarrow x$

ii) xRy & $yRx \Rightarrow x \Rightarrow y$.

iii) xRy & $yRz \Rightarrow x = z$.

partial Ordered Set (poset):

A set together with a partial order relation define on it is called partially ordered set or poset. It is denoted by $\leq$.

Eg:

1. Let R be the set of real numbers. The Relation $\leq$ is partial order of R. R is poset $(R, \leq)$

2. Let P(A) be the powerset of A. The relation $\subseteq$ (or Inclusio on P(A) is a partial order

$\therefore (P(A), \subseteq)$ is a poset.

Hasse diagram:

pictorial representation of a poset is called Hasse diagram.

1. Draw the Hasse diagram for $(P(A), \subseteq)$

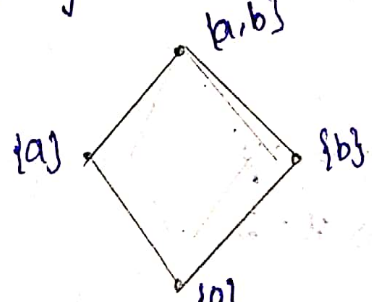
i) $A = \{a, b\}$ ii) $A = \{a, b, c\}$.

Solution:

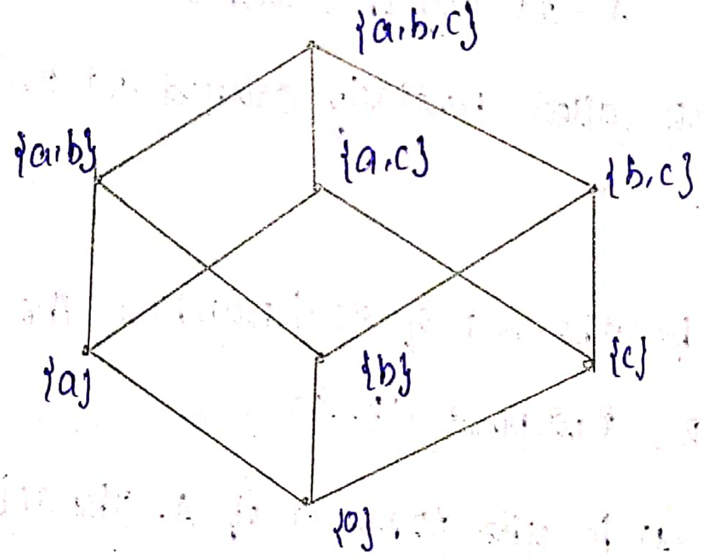
i) $P(A) = \{\{a,b\}, \{a\}, \{b\}, \{\emptyset\}\}$

The diagram can be represented as $(P(A), \subseteq)$

where $A = \{a, b\}$.



ii) $P(A) = \{\{a,b,c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a\}, \{b\}, \{c\}, \{\emptyset\}\}$.

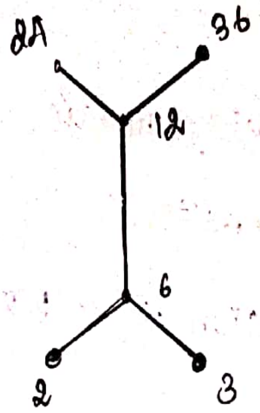


2. If $X = \{2, 3, 6, 12, 24, 36\}$ and the relation R defined on X by R.

$R = \{\langle a, b\rangle / a \mid b\}$.

Solution:

$R = \{\langle 2,6\rangle, \langle 2,12\rangle, \langle 2,24\rangle, \langle 2,36\rangle, \langle 3,6\rangle, \langle 3,12\rangle, \langle 3,24\rangle, \langle 3,36\rangle, \langle 6,12\rangle, \langle 6,24\rangle, \langle 6,36\rangle, \langle 12,24\rangle, \langle 12,36\rangle\}$.



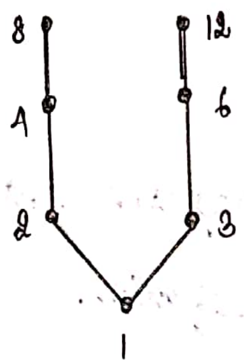
2. Draw the Hasse diagram for $\{(a,b) / a \text{ divides } b\}$.

i) $\{1, 2, 3, 4, 6, 8, 12\}$

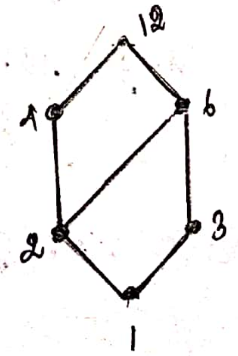
ii) $\{1, 2, 3, 4, 6, 12\}$.

Solution:

i) $R = \{(1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (2,4), (2,6), (2,8), (2,12), (3,6), (3,12); (4,8); (4,12), (6,12)\}$.



ii) $R = \{(1,2), (1,3), (1,4), (1,6), (1,12), (2,4), (2,6), (2,12), (3,6), (3,12), (4,12), (6,12)\}$.



Note:

i) $a/a \to$ Reflexive.

ii) a/b & $b/a \to a=b$ Antisymmetric

iii) a/b & $b/c \to a=c$ Transmittive

The divide relation is a partial order relation.

Theorem-1:

Show that $(N, \leq)$ is a partially order set, where N is the set of all +ve integers and $\leq$ defined by $m \leq n$ if and only if, n and m is a non-negative integer.

Solution:

Given that N is the set of all positive integers.

The relation $m \leq n$ if and only if $n-m$ is a non-negative integers

Now, $\forall\ x \in N$.

$x-x=0$ is a non-negative integers.

xRx, $\forall\ x \in N$. R is reflexive.

Consider,

$x \leq y$ and $y \leq x$

Since $xRy \Rightarrow x-y$ is a non-negative integer ...①.

$yRx \Rightarrow y-x = -(x-y)$ which is also a non-negative integer ...②.

From eqn ① and ②, we get

$x=y$.

$\therefore$ R is antisymmetric

Assume,

xRy and yRz

$xRy \Rightarrow x-y$ is a non-negative integer ... ③

$yRz \Rightarrow y-z$ is also a non-negative integer ... ④

Adding eqn ③ and ④.

$\Rightarrow x-y+y-z$ is a non-negative integer

$\Rightarrow x-z$ is a non-negative integer

$\Rightarrow xRz$.

xRy & $yRz \Rightarrow xRz$.

$\therefore$ R is transitive.

$\therefore (N, \leq)$ is a partial order relation

Least upper bound (LUB) / Supremum:

Let $(P, \leq)$ be a poset and $A \subseteq P$, an element $a \in P$ is said to be LUB if 'a' is a,

1) Upper bound of a.

2) $a \leq c$, where c is anyother upper bound of a.

Greatest Lower Bound (GLB) / Infimum:

Let $(P, \leq)$ be a poset and $A \subseteq P$, an element $b \in P$ is said to be GLB of 'a'. is

1) If a is lower bound. A.

2) $b \leq d$ where d is anyother greatest lower bound of A.

Eg: <u>Consider,</u>

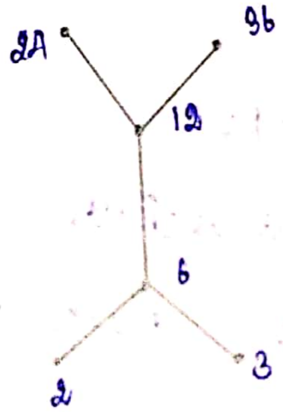
$X = \{2, 3, 6, 12, 24, 36\}$

$R = \{\langle a, b\rangle / a \mid b\}$.

1. Find LUB and GLB of (2, 3) and (24, 36)

<u>Solution:</u>

$R = \{\langle 2,6\rangle, \langle 2,12\rangle, \langle 2,24\rangle, \langle 2,36\rangle, \langle 3,6\rangle, \langle 3,12\rangle, \langle 3,24\rangle,$
$\langle 3,36\rangle, \langle 6,12\rangle, \langle 6,24\rangle, \langle 6,36\rangle, \langle 12,24\rangle, \langle 12,36\rangle\}$



i) <u>LUB:</u>

UB $\{2, 3\} \Rightarrow \{6, 12, 24, 36\}$.

LUB $\{2, 3\} \Rightarrow \{6\}$.

UB $\{24, 36\} \Rightarrow$ does not exist.

LUB $\{24, 36\}$ = does not exist.

ii) <u>GLB:</u>

LB $\{2, 3\} \Rightarrow$ does not exist.

GLB $\{2, 3\} \Rightarrow$ does not exist.

LB $\{24, 36\} \Rightarrow \{12, 6, 3, 2\}$.

GLB $\{24, 36\} \Rightarrow \{12\}$.

2. $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$ and let the relation be a partial ordering D_{24}.

i) Draw the Hasse diagram for D_{24} division.

ii) Find all LB of 8 and 12.

iii) Find all GLB of 8 and 12.

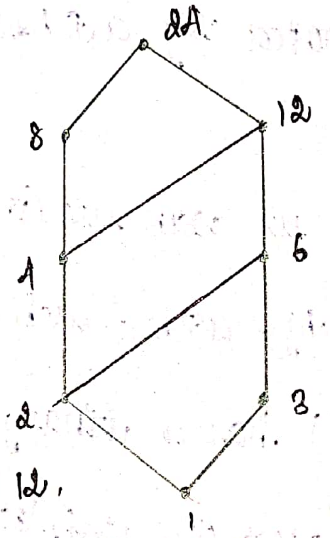
iv) Find all UB of 8 and 12.

v) Find LUB of 8 and 12.

vi) State the greatest and least element of the poset if it exists.

Solution:

i) Hasse diagram:



ii) The LB of 8 and 12.

LB $\{8, 12\} = \{4, 2, 1\}$.

iii) The GLB of 8 and 12.

GLB $\{8, 12\} = \{4\}$.

iv) The UB of 8 and 12.

UB $\{8, 12\} = \{24\}$.

v) The LUB of 8 and 12.

LUB $\{8, 12\} = \{24\}$.

vi) Greatest element of poset $\Rightarrow 24$.
Lowest element of poset $\Rightarrow 1$.

Lattice:

A lattice is a partially ordered set $(L, \leq)$ in which for every pair of elements $a, b \in L$. Both the greatest and lowest bound (GLB) and (LUB).

Note:

1. GLB $\{a, b\}$ is denoted by $a * b$. which is pronunced by 'a' meet 'b' (or) 'a' product 'b'.

Instead of $*$ we can use meet and dot ($\wedge$ or $\cdot$).

$\therefore$ GLB $\{a, b\} = a * b$ (or) $a \wedge b$ (or) $a \cdot b$.

2. LUB $\{a, b\}$ is denoted by $a \oplus b$ which is pronunced by 'a' joint 'b' (or) 'a' sum 'b'.

Instead of $\oplus$ we can use ($\vee$ and $+$)

$\therefore$ LUB $\{a, b\} = a \oplus b = a \vee b = a + b$.

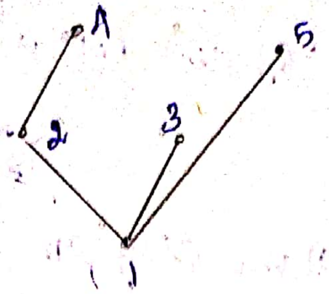
3. Since lattice $(L, \leq)$ has a binary operation $*$ ($\wedge$) and $+$ ($\vee$). a lattice can be denoted by triplet

$(L, *, \oplus)$, $(L, \wedge, \vee)$, $(L, \cdot, +)$.

1. Determine whether the poset
i) $\{\{1, 2, 3, 4, 5\}, /\}$ ii) $\{\{1, 2, 4, 8, 16\}, /\}$ are lattices.

Solution:

i) $R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 4)\}$.



LUB {2,3} does not exists.

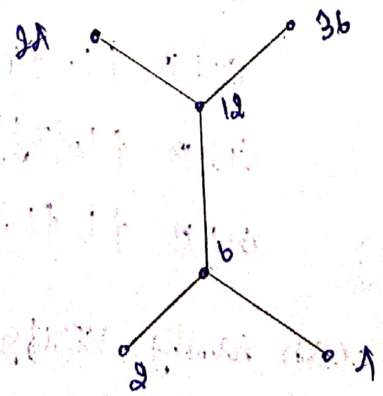
∴ The poset is not a lattice because it has no GLB and LUB.

ii) R = {(1,2), (1,4), (1,8), (1,16), (2,4), (2,8), (2,16), (4,8), (4,16), (8,16)}.

LUB {2,4} = 1.

Hence every pair of elements both GLB and LUB exists. ∴ the poset is lattice.

2. Determine the poset given by the Hasse diagram are lattice or not.



Solution:

Since LUB {21, 36} does not exists and GLB {2, 3} does not exists.

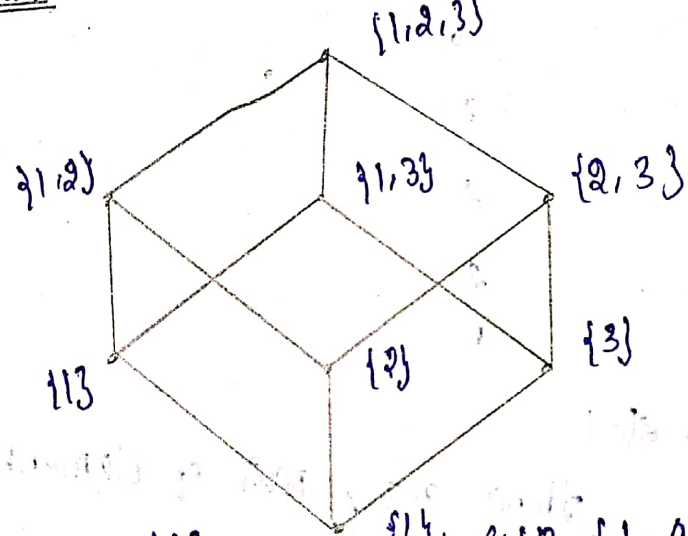
∴ The given Hasse diagram does not exists.

3. Determine whether $(P(A), \subseteq)$ is lattice $A = \{1, 2, 3\}$.

<u>Solution:</u>

$P(A) = \{\{1,2,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1\}, \{2\}, \{3\}, \{0\}\}$.

<u>Hasse diagram:</u>



LUB {1, 0} = {1}
LUB {1, {1,2}} = {1,2}
LUB {1, {1,3}} = {1,3}
LUB {1, {2,3}} = {1,2,3}
LUB {1, {2}} = {1,2}
LUB {1, {3}} = {1,3}
LUB {1, {1,2,3}} = {1,2,3}

GLB {1, 0} = {0}
GLB {1, {1,2}} = {1}
GLB {1, {1,3}} = {1}
GLB {1, {2,3}} = {0}
GLB {1, {2}} = {0}
GLB {1, {3}} = {0}
GLB {1, {1,2,3}} = {1}.

Similarly, we can easily verify both GLB and LUB exists for each pair of P(A). It is noticed that, for

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any two subsets A and B of $P(A)$.

$LUB\{A, B\} = A \vee B$. and

$GLB\{A, B\} = A \wedge B$.

which is exists.

$\therefore (P(A), \subseteq)$ is a lattice.

<u>properties of Lattices:</u>

Let $(L, \wedge, \vee)$ be a given lattice $\wedge, \vee$ satisifies

the condition. $\forall a, b, c \in L$.

1. <u>Idempotent law:</u>

$a \vee a = a$.

$a \wedge a = a$.

2. <u>Commutative law:</u>

$a \vee b = b \vee a$.

$a \wedge b = b \wedge a$.

3. <u>Associative law:</u>

$(a \vee b) \vee c = a \vee (b \vee c)$

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$

4. <u>Absorption law:</u>

$a \vee (a \wedge b) = a$

$a \wedge (a \vee b) = a$

<u>property 1:</u>

<u>Idempotent law:</u>

Let $(L, \wedge, \vee)$ be a given lattice. then $a, b, c \in L$,

$a \vee a = a$ and $a \wedge a = a$.

<u>proof:</u>

$a \vee a \Rightarrow LUB\{a,a\} = LUB\{a\} \Rightarrow a.$

$a \wedge a \Rightarrow GLB\{a,a\} = GLB\{a\} \Rightarrow a.$

Commutative law:

Let $(L, \wedge, \vee)$ be a given lattice and $a, b, c \in L$,

then prove $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.

proof:

$a \vee b \Rightarrow LUB\{a,b\} \Rightarrow LUB\{b,a\} \Rightarrow b \vee a.$

similarly,

$a \wedge b \Rightarrow GLB\{a,b\} \Rightarrow GLB\{b,a\} \Rightarrow b \wedge a.$

Absorption law:

Let $(L, \wedge, \vee)$ be a given lattice and $\forall a, b, c \in L$

then prove that, $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

proof:

Since $a \wedge b = GLB\{a,b\}$

$\Rightarrow a \wedge b \leq a \ldots$ ①

obviously, $a \leq a \ldots$ ②

By the eqn ① and ②

$a \vee (a \wedge b) \leq a \ldots$ ③

By the definition of LUB, we have

$a \leq a \vee (a \wedge b) \ldots$ ④

From ③ and ④

$a = a \vee (a \wedge b)$

$\therefore a \vee (a \wedge b) = a$

Similarly, $a \wedge (a \vee b) = a$.

Theorem-2:

Let $(L, \wedge, \vee)$ be a lattice, in which $\wedge$ and $\vee$ denotes the operation of $\wedge$ and $\vee$ respectively. For any $a, b \in L$, $a \le b$ if and only if $a \vee b = b$, if and only if $a \wedge b = a$ (or) $a \le b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a$.

Theorem-3:

State and prove distributive inequality of lattice.

Statement:

Let $(L, \wedge, \vee)$ be a given lattice. for any $a, b, c \in L$. The following inequality holds

i) $a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$

ii) $a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$.

proof:

1) $a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$.

From the defenition of LUB, it is obvious that,

$a \le a \vee b \ldots$ ①.

and $b \wedge c \le b \le a \vee b$

$\Rightarrow b \wedge c \le a \vee b \ldots$ ②

From ① and ②,

avb is a upper bound $\{a, b \wedge c\}$.

Hence, $a \vee b \geq a \vee (b \wedge c)$ .... Ⓐ

From the defenition it is obvious that,

$a \leq a \vee c$ ... ③

and $b \wedge c \leq c \leq a \vee c$

$\Rightarrow b \wedge c \leq a \vee c$ ... ④

From ③ and ④

avc is a upper bound $\{a, b \wedge c\}$.

Hence, $a \vee c \geq a \vee (b \wedge c)$ ... Ⓑ

From Ⓐ and Ⓑ

$a \vee (b \wedge c)$ is a lower bound of $(a \vee b), (a \vee c)$

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

Hence proved.

2) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

From the defenition of $a \wedge b$, it is obvious that,

$a \geq a \wedge b$ ... ①

and $b \vee c \geq b \geq a \wedge b$

$b \vee c \geq a \wedge b$ ... ②

From ① and ②,

$a \wedge b$ is a lower bound $\{a, b \vee c\}$.

$a \wedge b \leq a \wedge (b \vee c) \ldots$ Ⓐ

From the defenition, it is obvious that,

$a \geq a \wedge c \ldots$ ③

and $b \vee c \geq c \geq a \wedge c$.

$\Rightarrow b \vee c \geq a \wedge c \ldots$ ④

From ③ and ④

$a \wedge c$ is a lower bound $\{a, b \vee c\}$.

Hence

$a \wedge c \leq a \wedge (b \vee c) \ldots$ Ⓑ.

From Ⓐ and Ⓑ

$a \wedge (b \vee c)$ is a upper bound of $\{a \wedge b, a \wedge c\}$.

$\therefore a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$.

Hence proved.

Distributive lattice:

A lattice $(L, \wedge, \vee)$ is said to distributive if $\wedge$ and $\vee$ satisifies the following conditions.

$\forall$, $a, b, c \in L$

$D_1 \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$D_2 \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Theorem: 4
prove that any chain is a distributive lattice.

proof: Let $(L, \wedge, \vee)$ be a given chain and $\forall a, b, \in L$.

Since, any two elements of chain are compared is either

$a \leq b$ or $b \leq a$.

Case i): $a \leq b$

$LUB \{a, b\} = b$

$GLB \{a, b\} = a$.

Case ii) $b \leq a$.

$LUB \{b, a\} = a$

$GLB \{a, b\} = b$

In both cases, any two elements of a chain has both GLB and LUB.

$\therefore$ Any chain is a lattice.

Next we prove,

$(L, \wedge, \vee)$ satisfies distributive property.

Let $a, b, c, \in L$.

Since, any chain satisfies is a comparable property, we have the following six cases.

Case i): $a \leq b \leq c$.

Case ii): $a \leq c \leq b$.

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Case iii): $b \leq a \leq c$

case iv): $b \leq c \leq a$

case v): $c \leq b \leq a$

case vi): $c \leq a \leq b$

Case 1): $a \leq b \leq c$

Prove:

$D_1 \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

LHS:

$a \vee (b \wedge c)$.

$\Rightarrow a \vee (b \wedge c)$

$\Rightarrow a \vee b \quad [\because b \leq c,\ b \wedge c = b]$

$\Rightarrow b \quad [\because a \leq b,\ a \vee b = b]$

RHS: $(a \vee b) \wedge (a \vee c)$

$\Rightarrow b \wedge c \quad [\because a \leq b,\ a \leq c]$.

$\Rightarrow b$

$\therefore$ LHS = RHS.

$\therefore D_1$ condition is true for case 1.

Similarly,

we can easily prove the D_1 property for the remaining five cases.

$\therefore (L, \wedge, \vee)$ is a distributive lattice.

$\therefore$ Any chain is a distributive lattice.

Theorem-5 [Modular inequality]:

If $(L, \wedge, \vee)$ is a lattice, Then any a, b, c

$$a \leq c \Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c.$$

proof:

assume, $a \leq c$

By the definition of GLB & LUB we get

$$\Rightarrow a \wedge c = a \quad \cdots (1)$$

$$a \vee c = c \quad \cdots (2)$$

By distribution inequality we have,

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c) \quad \cdots (2)$$

using (1)

$$a \vee (b \wedge c) \leq (a \vee b) \wedge c \quad \cdots (A)$$

Conversly,

Assume $a \vee (b \wedge c) \leq (a \vee b) \wedge c$

Now by the defenition of LUB and GLB, we have

$$a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$$

$$\Rightarrow a \leq c \quad \cdots (B)$$

From (A) and (B)

$$a \leq c \Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$$

Hence Proved.

Modular Lattice:

A lattice $(L, \wedge, \vee)$ is said to be modular lattice, if it satisfies the following condition,

$$\text{If } \quad a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c.$$

Theorem-6:

Every distributive lattice is modular but not conversely.

Proof:

Let $(L, \wedge, \vee)$ be the given distributive lattice

$$D_1 \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \text{ holds good, } \forall a, b, c, \in L.$$

Now if, $a \leq c$ then $a \vee c \Rightarrow c \cdots$ ①

$$① \Rightarrow \quad a \vee (b \wedge c) \leq (a \vee b) \wedge c \cdots ②$$

There fore if $a \leq c \Leftrightarrow a \vee (b \wedge c) = (a \vee b) \wedge c.$

$\therefore$ every distributive lattice is Modular, but converse is not true.

Modular, but

That is every modular lattice need not be distributive.

1. If any distributive lattice $(L, \wedge, \vee)$, $\forall$ $a, b, c \in L$ prove that $a \vee b = a \vee c$, $a \wedge b = a \wedge c \Rightarrow b = c$

Solution:

$b = b \vee (b \wedge a)$ (absorption law)

$b \Rightarrow b \vee (a \wedge b)$ (Commutative law)

$b \Rightarrow b \vee (a \wedge c)$ (Since by given cond)

$\Rightarrow (b \vee a) \wedge (b \vee c)$ (Di law)

$b \Rightarrow (a \vee b) \wedge (b \vee c)$ (commutative law)

$b \Rightarrow (a \vee c) \wedge (b \vee c)$ (given cond)

$\Rightarrow (c \vee a) \wedge (c \vee b)$ (commutative law)

$\Rightarrow c \vee (a \wedge b)$ (Di $\rightarrow$ law)

$\Rightarrow c \vee (a \wedge c)$ (given cond)

$b \Rightarrow c \vee (c \wedge a)$ (commutative law)

$b \Rightarrow c$ (absorption law)

Theorem-4:

State and prove Isotonicity property.

Solution:

Let $(L, \wedge, \vee)$ be a given lattice.

For any $a, b, c \in L$.

We have,

$b \leq c$ i) $a \wedge b \leq a \wedge c$

ii) $a \vee b \leq a \vee c$.

Given,

$b \leq c$

$\therefore$ GLB of given $\{b, c\} \Rightarrow b \wedge c \Rightarrow b$ ... (1)

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$LUB\{b,c\} \Rightarrow b \vee c = c \cdots (2)$

claim 1):

$a \wedge b \leq a \wedge c$

It is enough to prove

$GLB\{a \wedge b, a \wedge c\} \Rightarrow (a \wedge b) \wedge (a \wedge c)$

$GLB\{a \wedge b, a \wedge c\} \Rightarrow a \wedge b$

LHS:

$(a \wedge b) \wedge (a \wedge c)$

$\Rightarrow a \wedge (b \wedge a) \wedge c$ (Associative law)

$\Rightarrow a \wedge (a \wedge b) \wedge c$ (Commutative law)

$\Rightarrow (a \wedge a) \wedge (b \wedge c)$ (Associative law)

$\Rightarrow a \wedge (b \wedge c)$ (Idempotent law)

$\Rightarrow a \wedge b$

$\Rightarrow$ RHS

claim 1 is proved.

claim 2:

$a \vee b \leq a \vee c$.

It is enough to prove

$GLB\{a \vee b, a \vee c\} \Rightarrow (a \vee b) \vee (a \vee c) \Rightarrow a \vee c$

LHS:

$(a \vee b) \vee (a \vee c)$

$\Rightarrow a \vee (b \vee a) \vee c$ (Associative law)

$\Rightarrow a \vee (a \vee b) \vee c$ (Commutative law)

$\Rightarrow (a \vee a) \vee (b \vee c)$ (associative law)

$\Rightarrow a \vee (b \vee c)$ (Idempotent law)

$\Rightarrow a \vee c$

$\Rightarrow$ RHS

claim 2 is proved.

Lattice as an algebraic System:

A lattice is an algebraic System $(L, \wedge, \vee)$ with two binary operations $\wedge$ and $\vee$ on L. which are both commutative, associative and satisfies absorption laws.

Sublattices:

Let $(L, \wedge, \vee)$ be a lattice and $S \subseteq L$ be a subset of L then $(S, \wedge, \vee)$ is a Sublattices of $(L, \wedge, \vee)$ if and only if S is closure under both operations $\wedge$ and $\vee$. $\forall$ $a, b \in S$ implies $a \wedge b \in S$ and $a \vee b \in S$.

Lattice Homomorphism:

Let $(L_1, \wedge, \vee)$ and $(L_2, *, \oplus)$ be two given lattice. A mapping $f: L_1 \rightarrow L_2$ is called lattice homomorphism if $\forall$, $a, b \in L$.

i) $f(a \wedge b) \Rightarrow f(a) * f(b)$

ii) $f(a \vee b) \Rightarrow f(a) \oplus f(b)$.

Ordered preserving:

A mapping from $L_1 \to L_2$ is said to be ordered preserving map from lattice $(L_1, \leq)$ to $(L_2, \leq)$ if $a \leq b$, then $f(a) \leq f(b)$.

Theorem-8:

Prove that any lattice homomorphism is order preserving.

Proof:

Let $f: L_1 \to L_2$ be a lattice homomorphism.

$a \leq b$, then the GLB of a, b is,

GLB $\{a, b\} \Rightarrow a \wedge b = a \ldots$ (1)

Then LUB $\{a, b\} \Rightarrow (a \vee b) = b \ldots$ (2)

Now, $f(a \wedge b) \Rightarrow f(a)$ using (1)

$f(a) \wedge f(b) \Rightarrow f(a)$ [Since f is homomorphism].

$\Rightarrow$ GLB $\{f(a), f(b)\} = f(a)$

$\Rightarrow f(a) \leq f(b)$.

$\therefore f$ is ordered preserving.

Note:

1. Least element is denoted by symbol '0' and it satisfies the condition, $0 \wedge a = 0$ and $0 \vee a = a$.

2. The greatest element is denoted by '1' and it satisfies the condition

$$1 \wedge a = a \text{ and } 1 \vee a = 1.$$

Complement:

Let $(L, \wedge, \vee, 0, 1)$ be given bounded lattice. Let 'a' be any element of L, we say that 'b' is complement of 'a'. If $a \wedge b = 0$ and $a \vee b = 1$ and 'b' is denoted by a symbol a' is

i.e, $a \wedge a' = 0$ and $a \vee a' \Rightarrow 1$.

Complemented Lattice:

A bounded lattice $(L, \wedge, \vee, 0, 1)$ is said to be complemented lattice, if every element of L has atleast one complement.

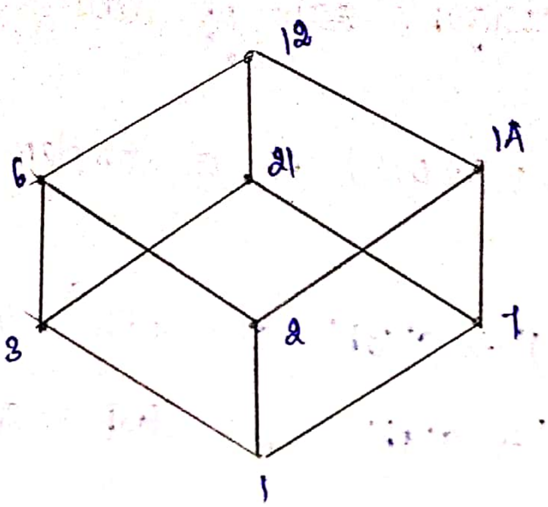
1. If S_{42} is the set of all divisions of 42 and D is relation divisor of on S_{42}. prove that $\{S_{42}, D\}$ is a complemented lattice.

Solution:

$S_{42} = \{$ All divisor of $42\}$

$S_{42} = \{1, 2, 3, 7, 14, 21, 42\}$

The Hasse diagram of (S_{42}, D) is



$0 =$ least element $\Rightarrow 1$.

$1 =$ greatest element $\Rightarrow 42$.

LUB $\{1, 42\} =$ LCM $\{1, 42\} = 42$

GLB $\{1, 42\} =$ GCD $\{1, 42\} = 1$.

$\therefore$ Complement of 1 is 42.

$\Rightarrow (1)' = 42$.

LUB $\{2, 21\} =$ LCM $\{2, 21\} = 42$

GLB $\{2, 21\} =$ GCD $\{2, 21\} = 1$.

$\therefore$ complement of 2 is 21.

$(2)' \Rightarrow 21$.

$(3)' = 14$.

$(6)' = 7$.

$(7)' = 6$.

$(14)' = 3$.

$(21)' = 2$.

$(42)' = 1$.

Since every element of S_{42} has complement.

$\therefore (S_{42}, D)$ is complemented lattice

Theorem-9: De Morgan's Law of Lattice

Statement: If $(L, \wedge, \vee, 0, 1)$ is a complemented lattice, then

prove that

i) $(a \wedge b)' \Rightarrow a' \vee b'$ (or) $\overline{a \wedge b} = \bar{a} \vee \bar{b}$

ii) $(a \vee b)' \Rightarrow a' \wedge b'$ (or) $\overline{a \vee b} = \bar{a} \wedge \bar{b}$.

Proof:

Claim i):

$(a \wedge b)' \Rightarrow a' \vee b'$.

It is enough to prove

1) $(a \wedge b) \wedge (a' \vee b') = 0$

2) $(a \wedge b) \vee (a' \vee b') = 1$.

1) $(a \wedge b) \wedge (a' \vee b')$

$\Rightarrow [(a \wedge b) \wedge a'] \vee [(a \wedge b) \wedge b']$ → Distributive law

$\Rightarrow [(b \wedge a) \wedge a'] \vee [(a \wedge b) \wedge b']$ → Commutative law

$\Rightarrow [b \wedge (a \wedge a')] \vee [a \wedge (b \wedge b')]$ → Associative law

$\Rightarrow [b \wedge 0] \vee [a \wedge 0]$

$\Rightarrow 0 \vee 0$

$\Rightarrow 0$.

ii) $(a \wedge b) \vee (a' \vee b')$

$\Rightarrow [(a' \vee b') \vee a] \wedge [(a' \vee b') \vee b]$ (Distributive law)

$\Rightarrow [a \vee (a' \vee b')] \wedge [b \vee [b' \vee a']]$ (Commutative law)

$\Rightarrow [(a \vee a') \vee b'] \wedge [(b \vee b') \vee a']$ (associative law)

$\Rightarrow [1 \vee b'] \wedge [1 \vee a']$

$\Rightarrow 1 \wedge 1$

$\Rightarrow 1$

$\therefore$ claim is proved.

claim ii):

$(a \vee b)' = a' \wedge b'$.

It is enough to prove

1. $(a \vee b) \wedge (a' \wedge b') = 0$
2. $(a \vee b) \vee (a' \wedge b') = 1$

i) $(a \vee b) \wedge (a' \wedge b')$:

$\Rightarrow [a \wedge (a' \wedge b')] \vee [b \wedge (a' \wedge b')]$ (Distributive law)

$\Rightarrow [a \wedge (a' \wedge b')] \vee b \wedge [b' \wedge a']$ (commutative law)

$\Rightarrow [(a \wedge a') \wedge b'] \vee [(b \wedge b') \wedge a']$ (associative law).

$\Rightarrow [0 \wedge b'] \vee [0 \wedge a']$

$\Rightarrow 0 \vee 0$

$\Rightarrow 0$.

ii) $(a \vee b) \vee (a' \wedge b')$.

$\Rightarrow [(a \vee b) \vee a'] \wedge [(a \vee b) \vee b']$ (Distributive law)

$\Rightarrow [(b \vee a) \vee a'] \wedge [(a \vee b) \vee b']$ (Commutative law)

$\Rightarrow [b \vee (a \vee a')] \wedge [(a \wedge (b) \vee b')]$ (associative law)

$\Rightarrow [b \vee 1] \wedge [a \wedge 1]$

$\Rightarrow 1 \wedge 1$

$\Rightarrow 1$.

$\therefore$ claim 2 is proved.

De-morgan's Law is proved.

Theorem-10:

prove that in a complemented distributive lattice, complement is unique (or) $(L, \wedge, \vee, 0, 1)$ is a distributive lattice then each element $x \in L$, has atmost one complement.

Solution:

Let us assume x and y are two complement.

To prove,

$x = y$

Since, x is a complement of 'a'.

$$\left.\begin{aligned} a \wedge x &= 0 \\ a \vee x &= 1 \end{aligned}\right\} \cdots ①$$

Since, y is a complement of 'a'.

$$\left.\begin{aligned} a \wedge y &= 0 \\ a \vee y &= 1 \end{aligned}\right\} \cdots ②$$

Now:

$x = x \vee 0$

$\Rightarrow x \vee (a \wedge y)$ Since by ②

$x \Rightarrow (x \vee a) \wedge (x \vee y)$ (distributive law)

$x \Rightarrow (a \vee x) \wedge (x \vee y)$ (commutative law)

$\Rightarrow 1 \wedge (x \vee y)$

$x \Rightarrow x \vee y \cdots$ Ⓐ

Similarly,

$y = y \vee 0$

$y = y \vee (a \wedge x)$ [by eau 1]

$y \Rightarrow (y \vee a) \wedge (y \vee x)$

$\Rightarrow (a \vee y) \wedge (y \vee x)$

$y \Rightarrow 1 \wedge (a \vee y) \ldots$

$y \Rightarrow x \vee y \ldots$ (B)

From equ (A) and (B)

$x = y$

$\therefore$ The complement is unique, in a complemented distributive lattice.

Theorem-11:

In a complemented distributive lattice, show that following are equivalent.

$a \leq b \Rightarrow a \wedge b' = 0 \Rightarrow a' \vee b \Rightarrow 1 \Rightarrow b' \leq a'$.

(or)

The following are equivalence

i) $a \leq b$ ii) $a \wedge b' = 0$ iii) $a' \vee b = 1$ iv) $b' \leq a'$.

Solution:

Since given lattice is complemented distributive lattice

$a \wedge a' = 0$

$a \vee a' = 1$.

proof (1) $\Rightarrow$ proof (2):

assume, $a \leq b \Rightarrow a \wedge a = a$,

$a \vee b = b$,

$a \wedge b' = (a \wedge b) \wedge b'$

$\Rightarrow a \wedge (b \wedge b')$

$\Rightarrow a \wedge 0$

$a \wedge b' \Rightarrow 0$.

proof ② ⇒ ③

Let $a \wedge b' \Rightarrow 0$

Taking complement on both sides

$(a \wedge b')' \Rightarrow 0'$

$a' \vee (b')' \Rightarrow 1$

$a' \vee b \Rightarrow 1$

proof ③ ⇒ proof ④

Let $a' \vee b \Rightarrow 1.$

Taking $\wedge b'$ on both sides

$(a' \vee b) \wedge b' = 1 \wedge b'$

$(a' \wedge b') \vee (b \wedge b') \Rightarrow 1 \wedge b'$

$(a' \wedge b') \vee 0 \Rightarrow 1 \wedge b'$

$a' \wedge b' \Rightarrow b'$

$a' \geq b'$

$\Rightarrow b' \leq a'.$

proof ④ ⇒ proof ①

Let $b' \leq a'$

$\Rightarrow a' \wedge b' \Rightarrow b'$

Taking complement on both sides,

$(a' \wedge b') \Rightarrow (b')'$

$(a')' \vee (b')' \Rightarrow b.$

$a \vee b \Rightarrow b$

$a \leq b.$

1. Show that a chain of 3 or more elements is not complemented.

Solution:

Let $(L, \wedge, \vee)$ be the given chain.

We know that, in a chain any 2 elements are comparable.

Let $0, a \& 1$ be any 3 element of $(L, \wedge, \vee)$ with 0 as the least element and 1 as the greatest element.

Now, $0 \leq a \leq 1$

$0 \wedge a \Rightarrow 0$ $\quad$ $a \wedge 1 \Rightarrow a$

$0 \vee a \Rightarrow a$ $\quad$ $a \vee 1 \Rightarrow 1$.

In both cases, a does not have any complement.

Hence, any chain with 3 or more element is not complemented.

Boolean Algebra:

A complemented distributive lattice is called Boolean algebra. A non empty set B with together on two binary operations on $(+, \cdot)$ on B. An unary operation on B and two distinct elements 0 and 1 are called Boolean algebra. If the following axioms satisfies a, b satisfies b.

1. Commutative law:

$$a+b = b+a$$

$$a \cdot b = b \cdot a$$

2. Associative law:

$$a+(b+c) = (a+b)+c.$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

3. Distributive law:

$$a+(b \cdot c) \Rightarrow (a+b) \cdot (a+c).$$

$$a \cdot (b+c) \Rightarrow (a \cdot b) + (a \cdot c)$$

4. Identity law:

There exists $0, 1 \in B$.

$$a+0 = a$$

$$a \cdot 1 = a.$$

5. Complement law:

For any, $a \in B$ there exists an element $a' \in B$,

Such that $a \cdot a' \Rightarrow 0.$

$$a + a' \Rightarrow 1.$$

Note: Boolean algebra is usually denoted by $(B, +, \cdot, 0, 1)$.

properties:

1. **Idempotent law:**

   1) $a \cdot a = a$, $\forall a \in B$.

   2) $a + a = a$

2. **Dominance law (Boundedness law):**

   1) $a \cdot 0 = 0$ $\forall a \in B$.

   2) $a + 1 = 1$

3. **Involution law:**

   $(a')' \Rightarrow a$ $\forall a \in B$.

4. In a Boolean algebra $0' \Rightarrow 1$ and $1' \Rightarrow 0$.

5. **Absorption law:**

   1. $a \cdot (a+b) = a$ $\forall a, b \in B$.

   2. $a + (a \cdot b) = a$

Theorem-12:

In a Boolean algebra, prove that following statements are equivalent.

1) $a+b = b$ 2) $a \cdot b = a$ 3) $a' + b = 1$, 4) $a \cdot b' = 0$.

Solution:

One way of prooving, the equivalence is true.

proof ① ⇒ ②

Let $a+b = b$.

Now $a \cdot b = a(a+b)$

$\Rightarrow a$ (absorption law)

proof ② ⇒ ③

Let $a \cdot b = a$

Now, $a' + b \Rightarrow (a \cdot b)' + b$

$\Rightarrow a' + (b' + b)$ (Demorgan's law)

$a' + b \Rightarrow a' + 1$ (Complement law)

$\Rightarrow (a \cdot 0)'$ (Demorgan's law)

$\Rightarrow 0'$

$a' + b \Rightarrow 1$

proof ③ ⇒ ④.

Let $a' + b \Rightarrow 1$.

Now, $a \cdot b' \Rightarrow 0$.

Taking complement on both the sides,

$(a' + b)' \Rightarrow (1)'$

$(a')' \cdot (b')' \Rightarrow 0$.

$a \cdot b' \Rightarrow 0$

proof ④ ⇒ ①.

Let $a \cdot b' \Rightarrow 0$.

Taking complement on both the sides,

$(a \cdot b')' \Rightarrow 0'$.

$(a)' + (b')' \Rightarrow 1$.

$a' + b \Rightarrow 1$.

Now, $a + b \Rightarrow (a+b) \cdot 1$ (Identity law)

$\Rightarrow (a+b)(a'+b)$

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$a+b \Rightarrow (b+a).(b+a')$ (Commutative law)

$a+b \Rightarrow b+(a.a')$ (Distributive law)

$a+b \Rightarrow b+0$

$a+b \Rightarrow b$

Hence proved.

1. Prove that D_{110}, the set of all positive divisors of the positive integer 110 is Boolean algebra and find all its subalgebra.

Solution:

$D_{110} = \{1, 2, 5, 10, 11, 22, 55, 110\}$.

Since, D satisfies reflexive, antisymmetric and Transmittive property.

D is a partial order relation on D_{110}.

$\therefore D_{110}$, D is poset.

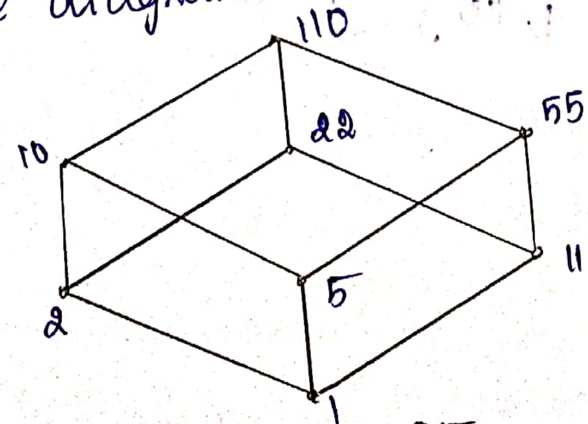
Here,

$a \wedge b = GCD\{a, b\}$

$a \vee b = LCM\{a, b\}$ $\forall$, $a, b \in D_{110}$.

$\therefore (D_{110}, \wedge, \vee)$ is a lattice.

Its Hasse diagram



Here, the least element (0) ⇒ 1

The greatest element (1) ⇒ 110.

Each and every element has its complement

Eg:

gcd $\{1, 110\} = 1$

lcm $\{1, 110\} = 110$

$(1)' = 110$ $\quad$ $(22)' = 5$

$(2)' = 55$ $\quad$ $(55)' = 2$

$(5)' = 22$ $\quad$ $(110)' = 1$

$(11)' = 10$ $\quad$ $(10)' = 11.$

∴ It is complemented lattice.

From the Hasse diagram,

It is obvious that, it is Distributive lattice.

∴ (D_{110}, D) is a Boolean algebra.

∴ The sub Boolean algebras are

i) $\{0, 1\}$ is $\{1, 110\}$

ii) $\{1, 2, 5, 10, 11, 22, 55, 110\}$.

iii) $\{a, a', 1, 110\}$ $\forall$ $a \in S$.

9. In a Boolean algebra show that $ab' + a'b = 0$ if and only if $a = b$.

Solution:

Let $a = b$,

Now, $ab' + a'b \Rightarrow aa' + a'a$

$\Rightarrow 0 + 0$

$\therefore ab' + a'b \Rightarrow 0$.

conversely,

assume that, $ab' + a'b \Rightarrow 0$

add 'a' on both sides,

$a + ab' + a'b \Rightarrow a$.

$a + a'b \Rightarrow a$ (absorption law)

$(a + a') \cdot (a + b) \Rightarrow a$ (Distributive law)

$1 \cdot (a + b) \Rightarrow a$.

$a + b \Rightarrow a$ ... (A)

similarly, $ab' + a'b \Rightarrow 0$.

add 'b' on both sides,

$ab' + a'b + b = b$.

$ab' + b = b$ (absorption law)

$(b + a) \cdot (b + b') \Rightarrow b$

$(b + a) \cdot 1 \Rightarrow b$ ... (B)

From eqn (A) and (B)

$a \Rightarrow b$. Hence proved.

3. Simplify the Boolean expression $a'b'c + a \cdot b'c + ab'c'$ using Boolean algebraic identities.

Solution:

$a'b'c + ab'c + ab'c'$

$\Rightarrow a'b'c + a \cdot b'(c + c')$.

$\Rightarrow a'b'c + a \cdot b'(1)$

$\Rightarrow (a'b')c + ab'$.

$\Rightarrow (b' \cdot a')c + (b' \cdot a)$ (Commutative law)

$\Rightarrow b'(a'c + a)$

$\Rightarrow b'(a + a'c)$ (Commutative law)

$\Rightarrow b'[(a + a') \cdot (a + c)]$ (Distributive law).

$\Rightarrow b'[1 \cdot (a + c)]$.

$\Rightarrow b'[a + c]$.

$\Rightarrow b'a + b'c$

$\therefore a'b'c + ab'c + ab'c' \Rightarrow ba' + b'c$

4. In any Boolean algebra. show that $(a + b')(b + c')(c + a') \Rightarrow (a'b)(b'+c)(c'+a)$.

Solution:

RHS: $(a + b')(b + c')(c + a')$

$\Rightarrow (a + b' + 0)(b + c' + 0)(c + a' + 0)$

$\Rightarrow$ ~~$(a + b' + cc')$~~ $(a + b' + cc')(b + c' + aa')(c + a' + bb')$

$\Rightarrow (a + b' + c)(a + b' + c')(b + c' + a)(b + c' + a')(c + a' + b)(c + a' + b')$

$\hookrightarrow$ distributive law

$\Rightarrow [(a'+b+c).(a'+b+c')].[(a+b'+c).(a'+b'+c)].[(a+b'+c').(a+b+c')]$.

$\Rightarrow (a'+b+cc').(b'+c+aa')(c'+a+bb')$ distributive law

$\Rightarrow (a'+b).(b'+c).(c'+a)$

$\Rightarrow$ RHS.

LHS = RHS.

Hence proved.

Theorem-13: Demorgan's law for Boolean algebra.

proof:

1. $(a.b)' = a'+b'$
2. $(a+b)' = a'.b'$

claim1: $(a.b)' = a'+b'$.

It is enough to prove that,

i) $(a.b).(a'+b') = 0$

ii) $(a.b)+(a'+b') = 1$

i) $(a.b).(a'+b')$.

$\Rightarrow [(a.b).a'] + [(a.b).b']$ distributive law

$\Rightarrow [(b.a).a'] + [(a.b).b']$ commutative law

$\Rightarrow [b.(a.a')] + [a.(b.b')]$ associative law

$\Rightarrow b.0 + a.0$

$\Rightarrow 0$.

ii) $(a.b)+(a'+b')$.

$\Rightarrow [(a'+b')+a].[(a'+b')+b]$ distributive law

$\Rightarrow [(b'+a')+a].[(a'+b')+b]$ commutative law

$\Rightarrow [(a'+a)+b'] \cdot [(b+b')+a']$ associative law

$\Rightarrow (1 \cdot b') \cdot (1 \cdot a')$

$\Rightarrow 1 \cdot 1$

$\Rightarrow 1$ $\therefore$ claim 1 is proved

claim 2): $(a+b)' \Rightarrow a' \cdot b'$.

It is enough to prove that,

i) $(a+b) \cdot (a' \cdot b') \Rightarrow 0$

ii) $(a+b)+(a' \cdot b') \Rightarrow 1$.

i) $(a+b) \cdot (a' \cdot b')$.

$\Rightarrow [(a' \cdot b') \cdot a] + [(a' \cdot b') \cdot b]$ (Distributive law)

$\Rightarrow [(b' \cdot a') \cdot a] + [(a' \cdot b') \cdot b]$ (Commutative law)

$\Rightarrow [b' \cdot (a' \cdot a)] + [a' \cdot (b' \cdot b)]$ (Associative law)

$\Rightarrow (b' \cdot 0) + (a' \cdot 0)$

$\Rightarrow 0+0$

$\Rightarrow 0$

ii) $(a+b)+(a' \cdot b')$.

$\Rightarrow [(a+b)+a'] \cdot [(a+b)+b']$ (Distributive law)

$\Rightarrow [(b+a)+a'] \cdot [(a+b)+b']$ (Commutative law)

$\Rightarrow [b+[a+a']] \cdot [a+(b+b')]$ (Associative law)

$\Rightarrow [b+1] \cdot [(1+a)]$

$\Rightarrow 1 \cdot 1$

$\Rightarrow 1$ $\therefore$ claim 2 is proved.

$\therefore$ Demorgan's law is verified.