UNIT V

LINEAR TIME INVARIANT-DISCRETE TIME SYSTEMS

Impulse response – Difference equations-Convolution sum- Discrete Fourier Transform and Z Transform Analysis of Recursive & Non-Recursive systems-DT systems connected in series and parallel.

6.6 Discrete Time System

A ***discrete time system*** is a device or algorithm that operates on a discrete time signal, called the input or excitation, according to some well defined rule, to produce another discrete time signal called the output or the response of the system. We can say that the input signal x(n) is transformed by the system into a signal y(n), and the transformation can be expressed mathematically as shown in equation (6.13).The diagrammatic representation of discrete time system is shown in fig 6.17.

$$\text{Response,} \quad y(n) = \mathcal{H}\{x(n)\} \qquad .....(6.13)$$

where, $\mathcal{H}$ denotes the transformation (also called an operator).

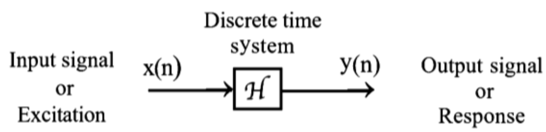


Fig 6.17 : *Representation of discrete time system.*

A discrete time system is linear if it obeys the principle of superposition and it is time invariant if its input-output relationship do not change with time. When a discrete time system satisfies the properties of linearity and time invariance then it is called an ***LTI system*** (Linear Time Invariant system).

Impulse Response

When the input to a discrete time system is a unit impulse δ(n) then the output is called an ***impulse response*** of the system and is denoted by h(n).

$$\therefore \text{ Impulse Response,} \quad h(n) = \mathcal{H}\{\delta(n)\} \qquad .....(6.14)$$

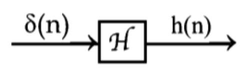


Fig 6.18 : *Discrete time system with impulse input.*

6.6.1 Mathematical Equation Governing Discrete Time System

The mathematical equation governing the discrete time system can be developed as shown below.

The response of a discrete time system at any time instant depends on the present input, past inputs and past outputs.

Let us consider the response at n = 0. Let us assume a relaxed system and so at n = 0, there is no past input or output. Therefore the response at n = 0, is a function of present input alone.

i.e., $y(0) = F[x(0)]$

Let us consider the response at n =1. Now the present input is x(1), the past input is x(0) and past output is y(0). Therefore the response at n = 1, is a function of x(1), x(0), y(0).

i.e., $y(1) = F[y(0),\ x(1),\ x(0)]$

Let us consider the response at n = 2. Now the present input is x(2), the past inputs are x(1) and x(0), and past outputs are y(1) and y(0). Therefore the response at n = 2, is a function of x(2), x(1), x(0), y(1), y(0).

i.e., $y(2) = F[y(1),\ y(0),\ x(2),\ x(1),\ x(0)]$

Similarly, at n = 3, $y(3) = F[y(2), y(1), y(0), x(3), x(2), x(1), x(0)]$

at n = 4, $y(4) = F[y(3), y(2), y(1), y(0), x(4), x(3), x(2), x(1)$, and so on.

In general, at any time instant n,

$$y(n) = F[y(n-1), y(n-2), y(n-3), \ldots y(1), y(0), x(n), x(n-1), x(n-2), x(n-3) \ldots x(1), x(0)] \qquad \ldots(6.15)$$

For an LTI system, the response y(n) can be expressed as a weighted summation of dependent terms. Therefore the equation (6.15) can be written as,

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - \ldots + b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + b_3\, x(n-3) + \ldots \qquad \ldots(6.16)$$

where, $a_1, a_2, a_3, \ldots$ and $b_0, b_1, b_2, b_3, \ldots$ are constants.

> ***Note :*** *Negative constants are inserted for output signals, because output signals are feedback from output to input. Positive constants are inserted for input signals, because input signals are feed forward from input to output.*

Practically, the response y(n) at any time instant n, may depend on N number of past outputs, present input and M number of past inputs where $M \le N$. Hence the equation (6.16) can be written as,

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - \ldots - a_N\, y(n-N) + b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + b_3\, x(n-3) + \ldots + b_M\, x(n-M)$$

$$\therefore \quad y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m) \qquad \ldots(6.17)$$

The equation (6.17) is a constant coefficient ***difference equation***, governing the input-output relation of an LTI discrete time system.

In equation (6.17) the value of "N" gives the ***order*** of the system.

If N = 1, the discrete time system is called 1st order system

If N = 2, the discrete time system is called 2nd order system

If N = 3, the discrete time system is called 3rd order system , and so on.

The general difference equation governing 1st order discrete time LTI system is,

$$y(n) = -a_1\, y(n-1) + b_0\, x(n) + b_1\, x(n-1)$$

The general difference equation governing 2nd order discrete time LTI system is,

$$y(n) = -a_2\, y(n-2) - a_1\, y(n-1) + b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2)$$

6.6.2 Block Diagram and Signal Flow Graph Representation of Discrete Time System

The discrete time system can be represented diagrammatically by ***block diagram*** or ***signal flow graph***. These diagrammatic representations are useful for physical implementation of discrete time system in hardware or software.

The basic elements employed in block diagram or signal flow graph are Adder, Constant multiplier, Unit delay element and Unit advance element.

Adder : An adder is used to represent addition of two discrete time sequences.

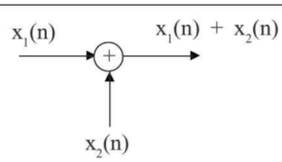
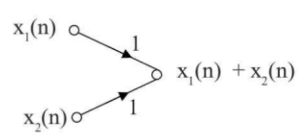
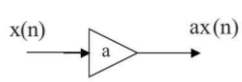
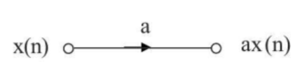
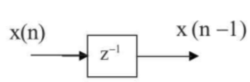
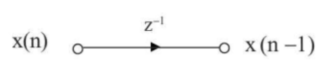
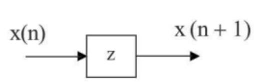
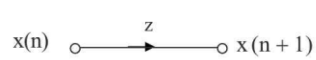
Constant Multiplier : A constant multiplier is used to represent multiplication of a scaling factor (constant) to a discrete time sequence.

Unit Delay Element : A unit delay element is used to represent the delay of samples of a discrete time sequence by one sampling time.

Unit Advance Element : A unit advance element is used to represent the advance of samples of a discrete time sequence by one sampling time.

The symbolic representation of the basic elements of block diagram and signal flow graph are listed in table 6.1.

Table 6.1 : Basic Elements of Block Diagram and Signal Flow Graph

Element	Block diagram representation	Signal flow graph representation
Adder		
Constant multiplier		
Unit delay element		
Unit advance element		

Example 6.7

Construct the block diagram and signal flow graph of the discrete time systems whose input-output relations are described by the following difference equations.

a) $y(n) = 0.5\ x(n) + 0.5\ x(n-1)$

b) $y(n) = 0.5\ y(n-1) + x(n) - 2\ x(n-2)$

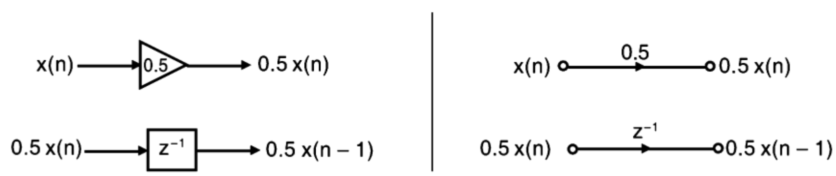
c) $y(n) = 0.25\ y(n-1) + 0.5\ x(n) + 0.75\ x(n-1)$

Solution

a) Given that, $y(n) = 0.5\ x(n) + 0.5\ x(n-1)$

The individual terms of the given equation are $0.5\ x(n)$ and $0.5\ x(n-1)$. They are represented by basic elements as shown below.

Block diagram representation | Signal flow graph representation



The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

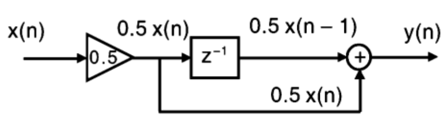


***Fig 1 :** Block diagram of the system*
$y(n) = 0.5\ x(n) + 0.5\ x(n-1)$.

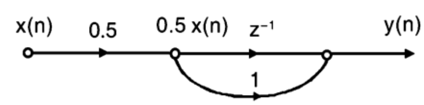


***Fig 2 :** Signal flow graph of the system*
$y(n) = 0.5\ x(n) + 0.5\ x(n-1)$.

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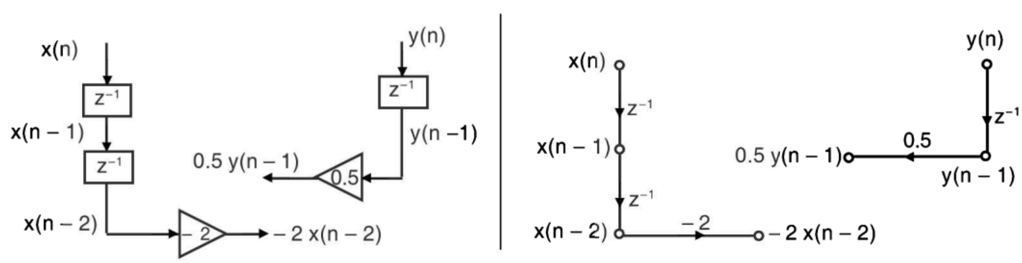
b) Given that, $y(n) = 0.5\ y(n-1) + x(n) - 2\ x(n-2)$

The individual terms of the given equation are $0.5\ y(n-1)$ and $-2\ x(n-2)$. They are represented by basic elements as shown below.

Block diagram representation | Signal flow graph representation



The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

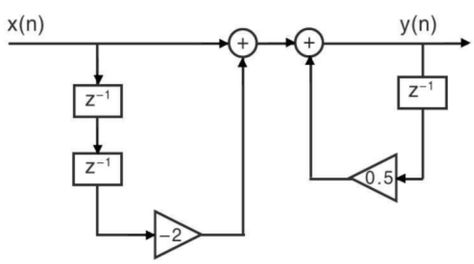


Fig 3 : *Block diagram of the system described by the equation*
$y(n) = 0.5\,y(n-1) + x(n) - 2\,x(n-2)$.

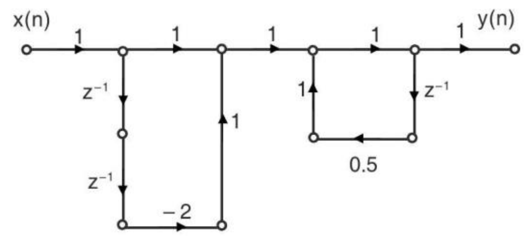
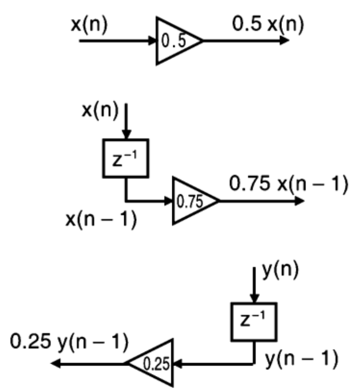


Fig 4 : *Signal flow graph of the system described by the equation*
$y(n) = 0.5\,y(n-1) + x(n) - 2\,x(n-2)$.

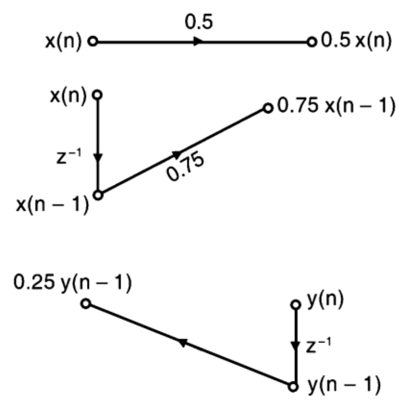
c) Given that, $y(n) = 0.25\,y(n-1) + 0.5\,x(n) + 0.75\,x(n-1)$

The individual terms of the given equation are 0.25 y(n – 1), 0.5 x(n) and 0.75 x(n – 1). They are represented by basic elements as shown below.

Block diagram representation



Signal flow graph representation



The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

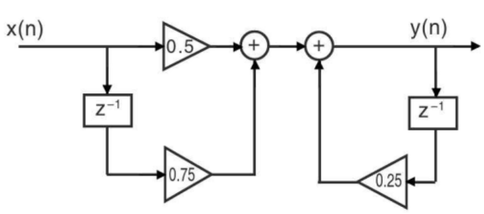


Fig 5 : *Block diagram of the system described by the equation*
$y(n) = 0.25\,y(n-1) + 0.5\,x(n) + 0.75\,x(n-1)$.

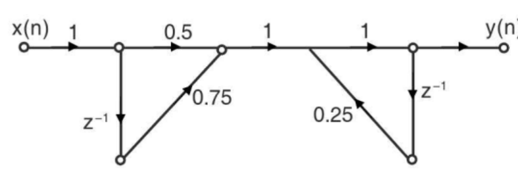


Fig 6 : *Signal flow graph of the system described by the equation*
$y(n) = 0.25\,y(n-1) + 0.5\,x(n) + 0.75\,x(n-1)$.

6.7 Response of LTI Discrete Time System in Time Domain

The general equation governing an LTI discrete time system is,

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\therefore \quad y(n) + \sum_{m=1}^{N} a_m\, y(n-m) = \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\text{(or)} \quad \sum_{m=0}^{N} a_m\, y(n-m) = \sum_{m=0}^{M} b_m\, x(n-m) \text{ with } a_o = 1 \qquad .....(6.18)$$

The solution of the difference equation (6.18) is the ***response*** $y(n)$ of LTI system, which consists of two parts. In mathematics, the two parts of the solution $y(n)$ are homogeneous solution $y_h(n)$ and particular solution $y_p(n)$.

$$\boxed{\therefore \text{ Response, } y(n) = y_h(n) + y_p(n)} \qquad .....(6.19)$$

The homogeneous solution is the response of the system when there is no input.The particular solution $y_p(n)$ is the solution of difference equation for specific input signal x(n) for $n \geq 0$.

In signals and systems, the two parts of the solution $y(n)$ are called zero-input response $y_{zi}(n)$ and zero-state response $y_{zs}(n)$.

$$\boxed{\therefore \text{ Response, } y(n) = y_{zi}(n) + y_{zs}(n)} \qquad .....(6.20)$$

The ***zero-input response*** is mainly due to initial conditions (or initial stored energy) in the system. Hence zero-input response is also called ***free response*** or ***natural response***. The ***zero-input response*** is given by homogeneous solution with constants evaluated using initial conditions.

The ***zero-state response*** is the response of the system due to input signal and with zero initial condition. Hence the zero-state response is called forced response.The ***zero-state response*** or ***forced response*** is given by the sum of homogeneous solution and particular solution with zero initial conditions.

6.7.1 Zero-Input Response or Homogeneous Solution

The ***zero-input response*** is obtained from homogeneous solution $y_h(n)$ with constants evaluated using initial condition.

$$\therefore \text{ Zero - input response, } y_{zi}(n) = y_h(n)\Big|_{\text{with constants evaluated using initial conditions}}$$

The ***homogeneous solution*** is obtained when x(n) = 0. Therefore the homogeneous solution is the solution of the equation,

$$\sum_{m=0}^{N} a_m\, y(n-m) = 0 \qquad .....(6.21)$$

Let us assume that the solution of equation (6.21) is in the form of an exponential.

i.e., $y(n) = \lambda^n$

On substituting $y(n) = \lambda^n$ in equation (6.21) we get,

$$\sum_{m=0}^{N} a_m \lambda^{n-m} = 0$$

On expanding the above equation (by taking $a_0 = 1$), we get,

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_{N-1} \lambda^{n-(N-1)} + a_N \lambda^{n-N} = 0$$
$$\lambda^{n-N} (\lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + \ldots + a_{N-1} \lambda + a_N) = 0$$

Now, the characteristic polynomial of the system is given by,

$$\lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + \ldots + a_{N-1} \lambda + a_N = 0$$

The characteristic polynomial has N roots, which are denoted as $\lambda_1, \lambda_2, \ldots \lambda_N$.

The roots of the characteristic polynomial may be distinct real roots, repeated real roots or complex. The assumed solutions for various types of roots are given below.

Distinct Real Roots

Let the roots $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_N$ be distinct real roots. Now the homogeneous solution will be in the form,

$$y_h(n) = C_1 \lambda_1^n + C_2 \lambda_2^n + C_3 \lambda_3^n + \ldots\ldots + C_N \lambda_N^n$$

where, $C_1, C_2, C_3, \ldots\ldots C_N$ are constants that can be evaluated using initial conditions.

Repeated Real Roots

Let one of the real roots λ_1 repeats p times and the remaining (N – p) roots are distinct real roots. Now, the homogeneous solution is in the form,

$$y_h(n) = (C_1 + C_2 n + C_3 n^2 + \ldots\ldots + C_p n^{p-1}) \lambda_1^n + C_{p+1} \lambda_{p+1}^n + \ldots\ldots + C_N \lambda_N^n$$

where, $C_1, C_2, C_3, \ldots\ldots C_N$ are constants that can be evaluated using initial conditions.

Complex Roots

Let the characteristic polynomial has a pair of complex roots λ and λ^* and the remaining (N – 2) roots be distinct real roots. Now, the homogeneous solution will be in the form,

$$y_h(n) = r^n [C_1 \cos n\theta + C_2 \sin n\theta] + C_3 \lambda_3^n + C_4 \lambda_4^n + \ldots + C_N \lambda_N^n$$

where, $\lambda = a + jb, \quad \lambda^* = a - jb, \quad r = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1} \frac{b}{a}$

$C_1, C_2, C_3 \ldots C_N$ are constants that can be evaluated using initial conditions.

6.7.2 Particular Solution

The ***particular solution***, $y_p(n)$ is the solution of the difference equation for specific input signal x(n) for $n \ge 0$. Since the input signal may have different form, the particular solution depends on the form or type of the input signal x(n).

If x(n) is constant, then $y_p(n)$ is also a constant.

Example :

Let, $x(n) = u(n)$; now, $y_p(n) = K\, u(n)$

If x(n) is exponential, then $y_p(n)$ is also an exponential.

Example :
Let, $x(n) = a^n u(n)$; now, $y_p(n) = K a^n u(n)$

If x(n) is sinusoid, then $y_p(n)$ is also a sinusoid.

Example :
Let, $x(n) = A \cos \omega_0 n$; now, $y_p(n) = K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$

The general form of particular solution for various types of inputs are listed in table 6.2.

Table 6.2 : Particular Solution

Input signal, x(n)	Particular solution, $y_p(n)$
A	K
AB^n	KB^n
An^B	$K_0 n^B + K_1 n^{(B-1)} + ... + K_B$
$A^n n^B$	$A^n (K_0 n^B + K_1 n^{(B-1)} + ...+ K_B)$
$\left.\begin{matrix} A \cos \omega_0 n \\ A \sin \omega_0 n \end{matrix}\right\}$	$K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$

6.7.3 Zero-State Response

The ***zero-state response*** or ***forced response*** is obtained from the sum of homogeneous solution and particular solution and evaluating the constants with zero initial conditions.

$$\therefore \text{ Zero - state response, } y_{zs}(n) = y_h(n) + y_p(n)\Big|_{\text{with constants } C_1,\, C_2, ... C_N \text{ evaluated with zero initial conditions}}$$

6.7.4 Total Response

The total response of discrete time system can be obtained by the following two methods.

Method-1

The ***total response*** is given by sum of homogeneous solution and particular solution.

$$\therefore \text{ Total response, } y(n) = y_h(n) + y_p(n)$$

Procedure to Determine Total Response by Method-1

1. Determine the homogeneous solution $y_h(n)$ with constants C_1, C_2, ...C_N.
2. Determine the particular solution $y_p(n)$ and evaluate the constants K for any value of $n \geq 1$ so that no term of y(n) vanishes.
3. Now the total response is given by the sum of $y_h(n)$ and $y_p(n)$.

   $$\therefore \text{ Total response, } y(n) = y_h(n) + y_p(n)$$
4. The total response will have N number of constants C_1, C_2, ...C_N. Evaluate the given equation and the total response for n = 0, 1, 2, ...N – 1 and form two sets of N number of equations and solve the constants C_1, C_2, ...C_N.

Method-2

The ***total response*** is given by sum of zero-input response and zero-state response.

$\therefore$ Total response, $y(n) = y_{zi}(n) + y_{zs}(n)$

Procedure to Determine Total Response by Method-2

1. Determine the homogeneous solution $y_h(n)$ with constants C_1, C_2, ...C_N.
2. Determine the zero-input response, which is obtained from the homogeneous solution $y_h(n)$ and evaluating the constants C_1, C_2, ...C_N using the initial conditions.
3. Determine the particular solution $y_p(n)$ and evaluate the constants K for any value of $n \geq 1$ so that no term of $y(n)$ vanishes.
4. Determine the zero-state response, $y_{zs}(n)$ which is given by sum of homogeneous solution and particular soulution and evaluating the constants C_1, C_2, ...C_N with zero initial conditions.
5. Now, the total response is given by sum of zero input response and zero state response.

   $\therefore$ Total response, $y(n) = y_{zi}(n) + y_{zs}(n)$

Example 6.8

Determine the response of first order discrete time system governed by the difference equation,

$y(n) = -0.5\, y(n-1) + x(n)$

When the input is unit step, and with initial condition a) $y(-1) = 0$ b) $y(-1) = 1/3$.

Solution

Given that, $y(n) = -0.5\, y(n-1) + x(n)$

$\therefore \; y(n) + 0.5\, y(n-1) = x(n)$ .....(1)

Homogeneous Solution

The homogeneous equation is the solution of equation (1) when $x(n) = 0$.

$\therefore \; y(n) + 0.5\, y(n-1) = 0$ .....(2)

Put, $y(n) = \lambda^n$ in equation (2).

$\therefore \; \lambda^n + 0.5\, \lambda^{(n-1)} = 0$

$\lambda^{(n-1)} (\lambda + 0.5) = 0 \quad \Rightarrow \quad \lambda = -0.5$

The homogeneous solution $y_h(n)$ is given by,

$y_h(n) = C\, \lambda^n = C\,(-0.5)^n \;;\quad \text{for } n \geq 0$ .....(3)

Particular Solution

Given that the input is unit step and so the particular solution will be in the form,

$y(n) = K\, u(n)$ .....(4)

On substituting for $y(n)$ from equation (4) in equation (1) we get,

$K\, u(n) + 0.5\, K\, u(n-1) = u(n)$ .....(5)

In order to determine the value of K, let us evaluate equation (5) for $n = 1$, ($\because$ we have to evaluate equation (5) for any $n \geq 1$, such that none of the term vanishes).

From equation (5) when n = 1, we get,

$$K + 0.5\,K = 1$$

$$1.5\,K = 1$$

$$\therefore \quad K = \frac{1}{1.5} = \frac{10}{15} = \frac{2}{3}$$

The particular solution $y_p(n)$ is given by,

$$y_p(n) = K\,u(n) = \frac{2}{3}\,u(n) \quad ; \quad \text{for all } n$$

$$= \frac{2}{3} \qquad ; \quad \text{for} \quad n \geq 0$$

Total Response

The total response y(n) of the system is given by sum of homogeneous and particular solution.

$\therefore$ Response, $y(n) = y_h(n) + y_p(n)$

$$= C(-0.5)^n + \frac{2}{3} \quad ; \quad \text{for } n \geq 0 \qquad .....(6)$$

At n = 0, from equation (1), we get, $y(0) + 0.5\,y(-1) = 1$

$$\therefore \quad y(0) = 1 - 0.5\,y(-1) \qquad ....(7)$$

At n = 0, from equation (6), we get, $\quad y(0) = C + \frac{2}{3} \qquad .....(8)$

On equating (7) and (8) we get, $\quad C + \frac{2}{3} = 1 - 0.5\,y(-1)$

$$\therefore\ C = 1 - 0.5\,y(-1) - \frac{2}{3}$$

$$= \frac{1}{3} - 0.5\,y(-1) \qquad .....(9)$$

On substituting for C from equation (9) in equation (6) we get,

$$y(n) = \left(\frac{1}{3} - 0.5\,y(-1)\right)(-0.5)^n + \frac{2}{3}$$

a) When y(−1) = 0

$$y(-1) = 0$$

$$\therefore\ y(n) = \frac{1}{3}(-0.5)^n + \frac{2}{3} \quad ; \quad \text{for} \quad n \geq 0$$

b) When y(−1) = 1/3

$$y(-1) = \frac{1}{3}$$

$$\therefore\ y(n) = \left(\frac{1}{3} - 0.5 \times \frac{1}{3}\right)(-0.5)^n + \frac{2}{3}$$

$$= \frac{0.5}{3}(-0.5)^n + \frac{2}{3}$$

$$= \frac{1}{6}(-0.5)^n + \frac{2}{3} \quad ; \quad \text{for } n \geq 0$$

Example 6.9

Determine the response y(n), $n \geq 0$ of the system described by the second order difference equation,

$$y(n) - 2\,y(n-1) - 3\,y(n-2) = x(n) + 4\,x(n-1),$$

when the input signal is, $x(n) = 2^n\,u(n)$ and with initial conditions $y(-2) = 0$, $y(-1) = 5$.

Solution

Given that, $y(n) - 2\,y(n-1) - 3\,y(n-2) = x(n) + 4\,x(n-1)$ .....(1)

Homogeneous Solution

The homogeneous equation is the solution of equation (1) when $x(n) = 0$.

$$\therefore\ y(n) - 2\,y(n-1) - 3\,y(n-2) = 0 \qquad .....(2)$$

Put $y(n) = \lambda^n$ in equation (2).

$$\therefore\ \lambda^n - 2\,\lambda^{n-1} - 3\,\lambda^{n-2} = 0$$

$$\lambda^{n-2}\,(\lambda^2 - 2\lambda - 3) = 0$$

The characteristic equation is,

$$\lambda^2 - 2\,\lambda - 3 = 0 \quad \Rightarrow \quad (\lambda - 3)\,(\lambda + 1) = 0$$

$\therefore$ The roots are, $\lambda = 3, -1$

The homogeneous solution, $y_h(n)$ is given by,

$$y_h(n) = C_1\,\lambda_1^n + C_2\,\lambda_2^n$$

$$= C_1(3)^n + C_2(-1)^n\ ; \quad \text{for } n \geq 0 \qquad .....(3)$$

Particular Solution

Given that the input is an exponential signal, $2^n\,u(n)$ and so the particular solution will be in the form,

$$y(n) = K\,2^n\,u(n) \qquad .....(4)$$

On substituting for y(n) from equation (4) in equation (1) we get,

$$K\,2^n\,u(n) - 2\,K\,2^{(n-1)}\,u(n-1) - 3\,K\,2^{(n-2)}\,u(n-2) = 2^n\,u(n) + 4 \times 2^{(n-1)}\,u(n-1) \qquad .....(5)$$

In order to determine the value of K, let us evaluate equation (5) for $n = 2$, ($\because$ we have to evaluate equation (5) for any $n \geq 1$, such that none of the term vanishes).

From equation (5) when $n = 2$, we get,

$$K\,2^2 - 2K \times 2^1 - 3K \times 2^0 = 2^2 + 4 \times 2^1$$

$$4K - 4K - 3K = 12$$

$$-3K = 12$$

$$\therefore\ K = -\frac{12}{3} = -4$$

The particular solution $y_p(n)$ is given by,

$$y_p\,(n) = K\,2^n\,u(n) = (-4)\,2^n\,u(n)$$

Total Response

The total response y(n) of the system is given by sum of homogeneous and particular solution.

$$\therefore\ \text{Response,}\ y(n) = y_h\,(n) + y_p\,(n)$$

$$= C_1\,3^n + C_2\,(-1)^n + (-4)\,2^n\ ; \quad \text{for } n \geq 0 \qquad .....(6)$$

When $n = 0$,

From equation (1) we get,

$$y(0) - 2\,y(-1) - 3\,y(-2) = x(0) + 4\,x(-1) \qquad .....(7)$$

Given that, $y(-1) = 5$, $y(-2) = 0$

$$x(n) = 2^n\,u(n), \quad \therefore\ x(0) = 2^0 = 1$$

$$x(-1) = 0$$

On substituting the above conditions in equation (7) we get,

$$y(0) - 2 \times 5 - 3 \times 0 = 1 + 0$$

$$\therefore\ y(0) = 11 \qquad .....(8)$$

When $n = 1$,

From equation (1) we get,

$$y(1) - 2\,y(0) - 3\,y(-1) = x(1) + 4\,x(0) \qquad .....(9)$$

We know that, $y(0) = 11$, $y(-1) = 5$, $y(-2) = 0$

Given that, $x(n) = 2^n\,u(n)$, $\therefore\ x(0) = 2^0 = 1$

$$x(1) = 2^1 = 2$$

On substituting the above conditions in equation (9) we get,

$$y(1) - 2 \times 11 - 3 \times 5 = 2 + 4 \times 1$$

$$\therefore\ y(1) = 6 + 37 = 43 \qquad .....(10)$$

When $n = 0$,

From equation (6) we get,

$$y(0) = C_1\,3^0 + C_2\,(-1)^0 + (-4)\,2^0 = C_1 + C_2 - 4 \qquad .....(11)$$

From equations (8) and (11) we can write,

$$C_1 + C_2 - 4 = 11$$

$$\therefore\ C_1 + C_2 = 15 \qquad .....(12)$$

When $n = 1$,

From equation (6) we get,

$$y(1) = C_1 \times 3 + C_2\,(-1) + (-4)\,2 = 3\,C_1 - C_2 - 8 \qquad .....(13)$$

From equations (10) and (13) we can write,

$$3\,C_1 - C_2 - 8 = 43$$

$$\therefore\ 3\,C_1 - C_2 = 51 \qquad .....(14)$$

On adding equations (12) and (14) we get,

$$4C_1 = 66$$

$$\therefore\ C_1 = \frac{66}{4} = \frac{33}{2}$$

From equation (12), $C_2 = 15 - C_1 = 15 - \dfrac{33}{2} = \dfrac{30-33}{2} = -\dfrac{3}{2}$

$$\therefore\quad y(n) = \frac{33}{2}\,(3)^n - \frac{3}{2}\,(-1)^n + (-4)\,2^n \quad ;\ \text{for } n \ge 0$$

$$= \left[\frac{33}{2}\,3^n - \frac{3}{2}\,(-1)^n - 4\,(2)^n\right] u(n)\ ;\ \text{for all n.}$$

6.8.6 FIR and IIR Systems

In ***FIR system*** (**F**inite duration **I**mpulse **R**esponse system), the impulse response consists of finite number of samples. The convolution formula for FIR system is given by,

$$y(n) = \sum_{m=0}^{N-1} h(m)\, x(n-m) \qquad .....(6.28)$$

where, $h(n) = 0$; for $n < 0$ and $n \geq N$

From equation (6.28) it can be concluded that the impulse response selects only N samples of the input signal.In effect, the system acts as a window that views only the most recent N input signal samples in forming the ouput. It neglects or simply forgets all prior input samples. Thus a FIR system requires memory of length N. In general, a FIR system is described by the difference equation,

$$y(n) = \sum_{m=0}^{N-1} b_m\, x(n-m)$$

where, $b_m = h(m)$; for $m = 0$ to $N-1$

In ***IIR system*** (**I**nfinite duration **I**mpulse **R**esponse system), the impulse response has infinite number of samples. The convolution formula for IIR systems is given by,

$$y(n) = \sum_{m=0}^{\infty} h(m)\, x(n-m)$$

Since this weighted sum involves the present and all the past input sample, we can say that the IIR system requires infinite memory. In general, an IIR system is described by the difference equation,

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

6.8.7 Recursive and Nonrecursive Systems

A system whose output y(n) at time n depends on any number of past output values as well as present and past inputs is called a ***recursive system.*** The past outputs are y(n−1), y(n−2), y(n−3), etc.,.

Hence for recursive system, the output y(n) is given by,

$$y(n) = F\,[y(n-1),\ y(n-2),\ldots y(n-N),\ x(n),\ x(n-1),\ldots x(n-M)]$$

A system whose output does not depend on past output but depends only on the present and past input is called a ***nonrecursive system.***

Hence for nonrecursive system, the output y(n) is given by,

$$y(n) = F\,[x(n),\ x(n-1)\ ,\ldots..,\ x(n-M)]$$

In a recursive system, in order to compute $y(n_0)$, we need to compute all the previous values y(0), y(1) ,......., $y(n_0-1)$ before calculating $y(n_0)$. Hence the output samples of a recursive system has to be computed in order [i.e., y(0), y(1), y(2), ...]. The IIR systems are recursive systems.

In nonrecursive system, $y(n_0)$ can be computed immediately without having $y(n_0-1)$, $y(n_0-2)$..... Hence the output samples of nonrecursive system can be computed in any order [i.e. y(50), y(5), y(2), y(100),....]. The FIR systems are nonrecursive systems.

6.9 Discrete or Linear Convolution

The ***Discrete*** or ***Linear convolution*** of two discrete time sequences $x_1(n)$ and $x_2(n)$ is defined as,

$$\boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)} \quad \text{or} \quad \boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_2(m)\, x_1(n-m)} \qquad .....(6.29)$$

where, $x_3(n)$ is the sequence obtained by convolving $x_1(n)$ and $x_2(n)$
m is a dummy variable

If the sequence $x_1(n)$ has N_1 samples and sequence $x_2(n)$ has N_2 samples then the output sequence $x_3(n)$ will be a finite duration sequence consisting of "N_1+N_2-1" samples. The convolution results in a nonperiodic sequence. Hence this convolution is also called ***aperiodic convolution.***

The convolution relation of equation (6.29) can be symbolically expressed as

$$x_3(n) = x_1(n) * x_2(n) = x_2(n) * x_1(n) \qquad ..... (6.30)$$

where, the symbol $*$ indicates convolution operation.

Procedure For Evaluating Linear Convolution

Let, $x_1(n)$ = Discrete time sequence with N_1 samples
$x_2(n)$ = Discrete time sequence with N_2 samples

Now, the convolution of $x_1(n)$ and $x_2(n)$ will produce a sequence $x_3(n)$ consisting of N_1+N_2-1 samples. Each sample of $x_3(n)$ can be computed using the equation (6.29). The value of $x_3(n)$ at n = q is obtained by replacing n by q, in equation (6.29).

$$\therefore\ x_3(q) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(q-m) \qquad .....(6.31)$$

The evaluation of equation (6.31) to determine the value of $x_3(n)$ at n = q, involves the following five steps.

1. Change of index : Change the index n in the sequences $x_1(n)$ and $x_2(n)$, to get the sequences $x_1(m)$ and $x_2(m)$.

2. Folding : Fold $x_2(m)$ about m = 0, to obtain $x_2(-m)$.

3. Shifting : Shift $x_2(-m)$ by q to the right if q is positive, shift $x_2(-m)$ by q to the left if q is negative to obtain $x_2(q-m)$.

4. Multiplication : Multiply $x_1(m)$ by $x_2(q-m)$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x_1(m) \times x_2(q-m)$.

5. Summation : Sum all the values of the product sequence $v_q(m)$ to obtain the value of $x_3(n)$ at n = q. [i.e., $x_3(q)$].

The above procedure will give the value $x_3(n)$ at a single time instant say n = q. In general, we are interested in evaluating the values of the seqence $x_3(n)$ over all the time instants in the range $-\infty < n < \infty$. Hence the steps 3, 4 and 5 given above must be repeated, for all possible time shifts in the range $-\infty < n < \infty$.

In the convolution of finite duration sequences it is possible to predict the start and end of the resultant sequence. If $x_1(n)$ starts at $n = n_1$ and $x_2(n)$ starts at $n = n_2$ then, the initial value of n for $x_3(n)$ is "$n = n_1 + n_2$". The value of $x_1(n)$ for $n < n_1$ and the value of $x_2(n)$ for $n < n_2$ are then assumed to be zero.The final value of n for $x_3(n)$ is "$n = (n_1 + n_2) + (N_1 + N_2 - 2)$".

6.9.1 Representation of Discrete Time Signal as Summation of Impulses

A discrete time signal can be expressed as summation of impulses and this concept will be useful to prove that the response of discrete time LTI system can be determined using discrete convolution.

Let, $x(n)$ = Discrete time signal

$\delta(n)$ = Unit impulse signal

$\delta(n-m)$ = Delayed impulse signal

We know that, $\delta(n) = 1$; at $n = 0$

$= 0$; when $n \neq 0$

and, $\delta(n-m) = 1$; at $n = m$

$= 0$; when $n \neq m$

If we multiply the signal $x(n)$ with the delayed impulse $\delta(n-m)$ then the product is non-zero only at $n = m$ and zero for all other values of n. Also at $n = m$, the value of product signal is m^{th} sample $x(m)$ of the signal $x(n)$.

$$\therefore\ x(n)\ \delta(n-m) = x(m)$$

Each multiplication of the signal $x(n)$ by an unit impulse at some delay m, in essence picks out the single value $x(m)$ of the signal $x(n)$ at $n = m$, where the unit impulse is non-zero. Consequently if we repeat this multiplication for all possible delays in the range $-\infty < m < \infty$ and add all the product sequences, the result will be a sequence that is equal to the sequence $x(n)$.

For example, $x(n)\ \delta(n-(-2)) = x(-2)$

$x(n)\ \delta(n-(-1)) = x(-1)$

$x(n)\ \delta(n) = x(0)$

$x(n)\ \delta(n-1) = x(1)$

$x(n)\ \delta(n-2) = x(2)$

From the above products we can say that each sample of $x(n)$ can be expressed as a product of the sample and delayed impulse, as shown below.

$\therefore\ x(-2) = x(-2)\ \delta(n-(-2))$

$x(-1) = x(-1)\ \delta(-(-1))$

$x(0) = x(0)\ \delta(n)$

$x(1) = x(1)\ \delta(n-1)$

$x(2) = x(2)\ \delta(n-2)$

$$\therefore x(n) = \cdots + x(-2) + x(-1) + x(0) + x(1) + x(2) + \cdots\cdots$$

$$= \cdots + x(-2)\ \delta(n - (-2)) + x(-1)\ \delta(n - (-1)) + x(0)\ \delta(n) + x(1)\ \delta(n-1) + x(2)\ \delta(n-2) + \cdots\cdots$$

$$= \sum_{m=-\infty}^{+\infty} x(m)\ \delta(n-m) \qquad .....(6.32)$$

In equation (6.32) each product $x(m)\ \delta(n-m)$ is an impulse and the summation of impulses gives the sequence $x(n)$.

6.9.2 Response of LTI Discrete Time System Using Discrete Convolution

In an LTI system, the response $y(n)$ of the system for an arbitrary input $x(n)$ is given by convolution of input $x(n)$ with impulse response $h(n)$ of the system. It is expressed as,

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\ h(n-m) \qquad .....(6.33)$$

where, the symbol $*$ represents convolution operation.

Proof :

Let $y(n)$ be the response of system $\mathcal{H}$ for an input $x(n)$

$$\therefore y(n) = \mathcal{H}\{x(n)\} \qquad .....(6.34)$$

From equation (6.32) we know that the signal $x(n)$ can be expressed as a summation of impulses,

$$\text{i.e., } x(n) = \sum_{m=-\infty}^{+\infty} x(m)\ \delta(n-m) \qquad .....(6.35)$$

where, $\delta(n-m)$ is the delayed unit impulse signal.

From equation (6.34) and (6.35) we get,

$$y(n) = \mathcal{H}\left\{\sum_{m=-\infty}^{+\infty} x(m)\ \delta(n-m)\right\} \qquad .....(6.36)$$

The system $\mathcal{H}$ is a function of n and not a function of m. Hence by linearity property the equation (6.36) can be written as,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\ \mathcal{H}\{\delta(n-m)\} \qquad .....(6.37)$$

Let the response of the LTI system to the unit impulse input $\delta(n)$ be denoted by $h(n)$,

$$\therefore h(n) = \mathcal{H}\{\delta(n)\}$$

Then by time invariance property the response of the system to the delayed unit impulse input $\delta(n-m)$ is given by,

$$h(n-m) = \mathcal{H}\{\delta(n-m)\} \qquad .....(6.38)$$

Using equation (6.38), the equation (6.37) can be expressed as,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\ h(n-m)$$

The above equation represents the convolution of input $x(n)$ with the impulse response $h(n)$ to yield the output $y(n)$. Hence it is proved that the response $y(n)$ of LTI discrete time system for an arbitrary input $x(n)$ is given by convolution of input $x(n)$ with impulse response $h(n)$ of the system.

6.9.3 Properties of Linear Convolution

The Discrete convolution will satisfy the following properties.

Commutative property : $x_1(n) * x_2(n) = x_2(n) * x_1(n)$

Associative property : $[x_1(n) * x_2(n)] * x_3(n) = x_1(n) * [x_2(n) * x_3(n)]$

Distributive property : $x_1(n) * [x_2(n) + x_3(n)] = [x_1(n) * x_2(n)] + [x_1(n) * x_3(n)]$

Proof of Commutative Property :

Consider convolution of $x_1(n)$ and $x_2(n)$.

By commutative property we can write,

$$\underset{\text{(LHS)}}{x_1(n) * x_2(n)} = \underset{\text{(RHS)}}{x_2(n) * x_1(n)}$$

$$\text{LHS} = x_1(n) * x_2(n)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m) \qquad .....(6.39)$$

where, m is a dummy variable used for convolution operation.

Let, $n - m = p$ | when $m = -\infty$, $p = n - m = n + \infty = +\infty$

$\therefore m = n - p$ | when $m = +\infty$, $p = n - m = n - \infty = -\infty$

On replacing m by $(n - p)$ and $(n - m)$ by p in equation (6.39) we get,

$$\text{LHS} = \sum_{p=-\infty}^{+\infty} x_1(n-p)\, x_2(p)$$

$$= \sum_{p=-\infty}^{+\infty} x_2(p)\, x_1(n-p)$$

$$= x_2(n) * x_1(n)$$

p is a dummy variable used for convolution operation

$$= \text{RHS}$$

Proof of Associative Property :

Consider the discrete time signals $x_1(n)$, $x_2(n)$ and $x_3(n)$. By associative property we can write,

$$\underset{\text{LHS}}{[x_1(n) * x_2(n)] * x_3(n)} = \underset{\text{RHS}}{x_1(n) * [x_2(n) * x_3(n)]}$$

Let, $y_1(n) = x_1(n) * x_2(n)$ .....(6.40)

Let us replace n by p

$$\therefore y_1(p) = x_1(p) * x_2(p)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(p-m) \qquad .....(6.41)$$

Let, $y_2(n) = x_2(n) * x_3(n)$ .....(6.42)

$$\therefore y_2(n) = \sum_{q=-\infty}^{+\infty} x_1(q)\, x_2(n-q)$$

$$\therefore\ y_2(n-m) = \sum_{q=-\infty}^{+\infty} x_1(q)\, x_2(n-q-m) \qquad .....(6.43)$$

where p, m and q are dummy variables used for convolution operation.

$$\text{LHS} = [x_1(n) * x_2(n)] * x_3(n)$$

$$= y_1(n) * x_3(n) \qquad \boxed{\text{Using equation (6.40)}}$$

$$= \sum_{p=-\infty}^{+\infty} y_1(p)\, x_3(n-p)$$

$$= \sum_{p=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(p-m)\, x_3(n-p) \qquad \boxed{\text{Using equation (6.41)}}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m) \sum_{p=-\infty}^{+\infty} x_2(p-m)\, x_3(n-p) \qquad .....(6.44)$$

Let, $p - m = q$ | when $p = -\infty$, $q = p - m = -\infty - m = -\infty$

$\therefore p = q + m$ | when $p = +\infty$, $q = p - m = +\infty - m = +\infty$

On replacing $(p - m)$ by q, and p by $(q + m)$ in the equation (6.44) we get,

$$\text{LHS} = \sum_{m=-\infty}^{+\infty} x_1(m) \sum_{q=-\infty}^{+\infty} x_2(q)\, x_3(n-q-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, y_2(n-m) \qquad \boxed{\text{Using equation (6.43)}}$$

$$= x_1(n) * y_2(n)$$

$$= x_1(n) * [x_2(n) * x_3(n)] \qquad \boxed{\text{Using equation (6.42)}}$$

$$= \text{RHS}$$

Proof of Distributive Property :

Consider the discrete time signals $x_1(n)$, $x_2(n)$ and $x_3(n)$. By distributive property we can write,

$$\underbrace{x_1(n) * [x_2(n) + x_3(n)]}_{\text{LHS}} = \underbrace{[x_1(n) * x_2(n)] + [x_1(n) * x_3(n)]}_{\text{RHS}}$$

$$\text{LHS} = x_1(n) * [x_2(n) + x_3(n)]$$

$$= x_1(n) * x_4(n) \qquad \boxed{x_4(n) = x_2(n) + x_3(n)}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_4(n-m) \qquad \boxed{\text{m is a dummy variable used for convolution operation}}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, [x_2(n-m) + x_3(n-m)]$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m) + \sum_{m=-\infty}^{+\infty} x_1(m)\, x_3(n-m)$$

$$= [x_1(n) * x_2(n)] + [x_1(n) * x_3(n)]$$

$$= \text{RHS}$$

6.9.4 Interconnections of Discrete Time Systems

Smaller discrete time systems may be interconnected to form larger systems. Two possible basic ways of interconnection are ***cascade connection*** and ***parallel connection***. The cascade and parallel connections of two discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ are shown in fig 6.21.

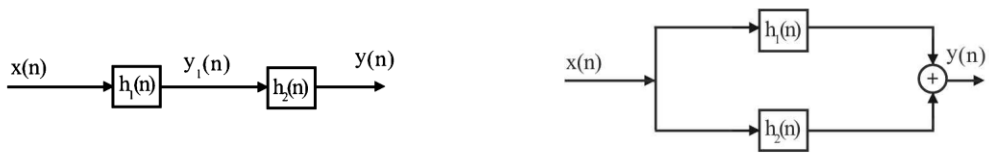


Fig 6.21a : *Cascade connection.* ***Fig 6.21b :*** *Parallel connection.*

Fig 6.21 : *Interconnection of discrete time systems.*

Cascade Connected Discrete Time System

Two cascade connected discrete time systems with impulse response $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system whose impulse response is given by convolution of individual impulse responses.

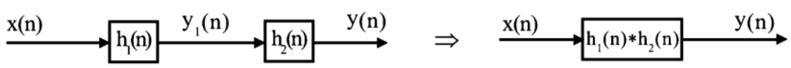


Fig 6.22 : *Cascade connected discrete time systems and their equivalent.*

Proof:

With reference to fig 6.22 we can write,

$$y_1(n) = x(n) * h_1(n) \qquad .....(6.45)$$

$$y(n) = y_1(n) * h_2(n) \qquad .....(6.46)$$

Using equation (6.45), the equation (6.46) can be written as,

$$y(n) = x(n) * h_1(n) * h_2(n)$$

$$= x(n) * [h_1(n) * h_2(n)]$$

$$= x(n) * h(n) \qquad .....(6.47)$$

where, $h(n) = h_1(n) * h_2(n)$

From equation (6.47) we can say that the overall impulse response of two cascaded discrete time systems is given by convolution of individual impulse responses.

Parallel Connected Discrete Time Systems

Two parallel connected discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system whose impulse response is given by sum of individual impulse responses.

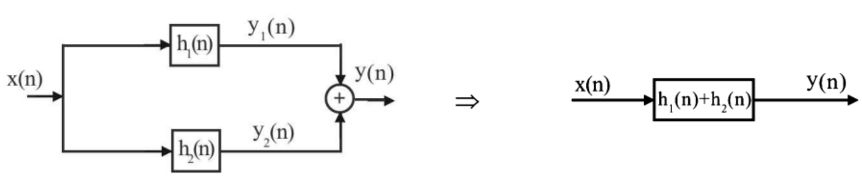


Fig 6.23 : *Parallel connected discrete time systems and their equivalent.*

Proof:

With reference to fig 6.23 we can write,

$y_1(n) = x(n) * h_1(n)$ .....(6.48)

$y_2(n) = x(n) * h_2(n)$ .....(6.49)

$y(n) = y_1(n) + y_2(n)$ .....(6.50)

On substituting for $y_1(n)$ and $y_2(n)$ from equations (6.48) and (6.49) in equation (6.50) we get,

$y(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)]$ .....(6.51)

By using distributive property of convolution the equation (6.51) can be written as shown below,

$$y(n) = x(n) * [h_1(n) + h_2(n)]$$

$$= x(n) * h(n) \quad .....(6.52)$$

where, $h(n) = h_1(n) + h_2(n)$

From equation (6.52) we can say that the overall impulse response of two parallel connected discrete time system is given by sum of individual impulse responses.

Example 6.19

Determine the impulse reponse for the cascade of two LTI systems having impulse responses,

$$h_1(n) = \left(\frac{1}{2}\right)^n u(n) \text{ and } h_2(n) = \left(\frac{1}{4}\right)^n u(n).$$

Solution

Let h(n) be the impulse response of the cascade system. Now h(n) is given by convolution of $h_1(n)$ and $h_2(n)$.

$$\therefore h(n) = h_1(n) * h_2(n)$$

$$= \sum_{m=-\infty}^{+\infty} h_1(m)\, h_2(n-m) \quad \text{where, m is a dummy variable used for convolution operation}$$

The product $h_1(m)\, h_2(n-m)$ will be non-zero in the range $0 \le m \le n$. Therefore the summation index in the above equation is changed to m = 0 to n.

$$\therefore h(n) = \sum_{m=0}^{n} h_1(m)\, h_2(n-m) = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{4}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{4}\right)^n \left(\frac{1}{4}\right)^{-m} = \left(\frac{1}{4}\right)^n \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m 4^m$$

$$= \left(\frac{1}{4}\right)^n \sum_{m=0}^{n} \left(\frac{4}{2}\right)^m$$

Finite geometric series sum formula

$$\sum_{n=0}^{N} C^n = \frac{C^{N+1}-1}{C-1}$$

$$= \left(\frac{1}{4}\right)^n \sum_{m=0}^{n} 2^m$$

$$= \left(\frac{1}{4}\right)^n \left(\frac{2^{n+1}-1}{2-1}\right)$$

Using finite geometric series sum formula

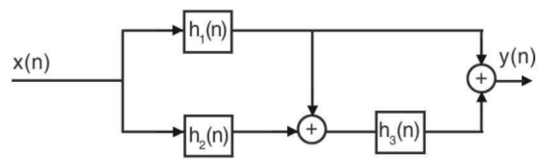
$$= \left(\frac{1}{4}\right)^n (2^{n+1}-1) \;;\; \text{for } n \ge 0$$

$$= \left(\frac{1}{4}\right)^n (2^{n+1}-1)\, u(n) \;;\; \text{for all } n$$

Example 6.20

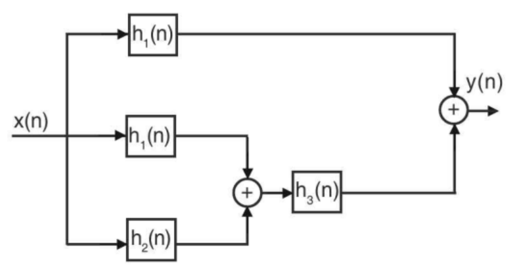
Determine the overall impulse response of the interconnected discrete time system shown below,

where, $h_1(n) = \left(\frac{1}{3}\right)^n u(n)$, $h_2(n) = \left(\frac{1}{2}\right)^n u(n)$ and $h_3(n) = \left(\frac{1}{5}\right)^n u(n)$.

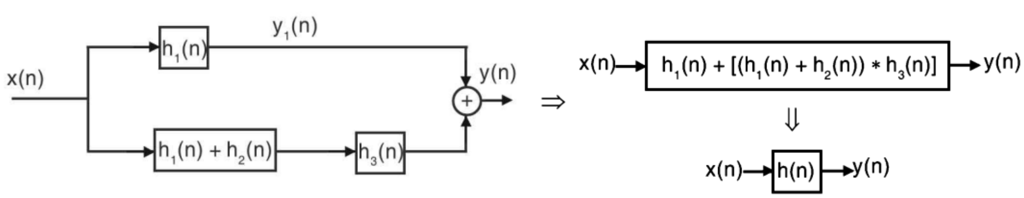


Solution

The given system can be redrawn as shown below.



The above system can be reduced to single equivalent system as shown below.



Here, $h(n) = h_1(n) + [(h_1(n) + h_2(n)) * h_3(n)]$

$= h_1(n) + [h_1(n) * h_3(n)] + [h_2(n) * h_3(n)]$ Using distributive property

Let us evaluate the convolution of $h_1(n)$ and $h_3(n)$.

$$h_1(n) * h_3(n) = \sum_{m=-\infty}^{\infty} h_1(m)\, h_3(n-m)$$

The product of $h_1(m)\, h_3(n-m)$ will be non-zero in the range $0 \le m \le n$. Therefore the summation index in the above equation can be changed to $m = 0$ to n.

$$\therefore\ h_1(n) * h_3(n) = \sum_{m=0}^{n} h_1(m)\, h_3(n-m)$$

$$= \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-m}$$

$$= \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m 5^m = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{5}{3}\right)^m$$

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{3}\right)^{n+1} - 1}{\frac{5}{3} - 1}$$

Using finite geometric series sum formula

Finite geometric series sum formula

$$\sum_{m=0}^{N} C^m = \frac{C^{N+1} - 1}{C - 1}$$

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{3}\right)^n \frac{5}{3} - 1}{\frac{5-3}{3}} = \left(\frac{1}{5}\right)^n \left[\frac{3}{2}\left(\frac{5}{3}\right)^n \frac{5}{3} - \frac{3}{2}\right]$$

$$= \frac{5}{2}\left(\frac{1}{5}\right)^n \left(\frac{5}{3}\right)^n - \frac{3}{2}\left(\frac{1}{5}\right)^n = \frac{5}{2}\left(\frac{1}{3}\right)^n - \frac{3}{2}\left(\frac{1}{5}\right)^n \text{ ; for } n \geq 0$$

$$= \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n) \text{ ; for all } n$$

Let us evaluate the convolution of $h_2(n)$ and $h_3(n)$.

$$h_2(n) * h_3(n) = \sum_{m=-\infty}^{+\infty} h_2(m)\, h_3(n-m)$$

The product of $h_2(m)$ and $h_3(n-m)$ will be non-zero in the range $0 \leq m \leq n$. Therefore the summation index in the above equation can be change to m = 0 to n.

$$\therefore\ h_2(n) * h_3(n) = \sum_{m=0}^{n} h_2(m)\, h_3(n-m)$$

$$= \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{5}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-m}$$

Finite geometric series sum formula

$$\sum_{m=0}^{N} C^m = \frac{C^{N+1} - 1}{C - 1}$$

$$= \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m 5^m = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{5}{2}\right)^m$$

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{2}\right)^{n+1} - 1}{\frac{5}{2} - 1}$$

Using finite geometric series sum formula

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{2}\right)^n \frac{5}{2} - 1}{\frac{5-2}{2}} = \left(\frac{1}{5}\right)^n \left[\frac{2}{3}\left(\frac{5}{2}\right)^n \frac{5}{2} - \frac{2}{3}\right]$$

$$= \frac{5}{3}\left(\frac{1}{5}\right)^n \left(\frac{5}{2}\right)^n - \frac{2}{3}\left(\frac{1}{5}\right)^n = \frac{5}{3}\left(\frac{1}{2}\right)^n - \frac{2}{3}\left(\frac{1}{5}\right)^n \text{ for } n \geq 0$$

$$= \frac{5}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{5}\right)^n u(n) \text{ for all } n$$

Now, the overall impulse response h(n) is given by,

$$h(n) = h_1(n) + \left[h_1(n) * h_3(n)\right] + \left[h_2(n) * h_3(n)\right]$$

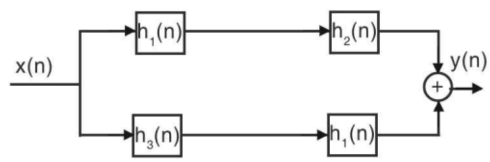
$$= \left(\frac{1}{3}\right)^n u(n) + \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{5}\right)^n u(n)$$

$$= \left(1 + \frac{5}{2}\right)\left(\frac{1}{3}\right)^n u(n) - \left(\frac{3}{2} + \frac{2}{3}\right)\left(\frac{1}{5}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{2}\right)^n u(n)$$

$$= \left[\frac{7}{2}\left(\frac{1}{3}\right)^n - \frac{13}{6}\left(\frac{1}{5}\right)^n + \frac{5}{3}\left(\frac{1}{2}\right)^n\right] u(n)$$

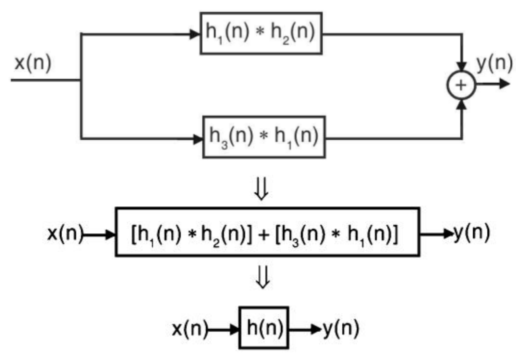
Example 6.21

Find the overall impulse response of the interconnected system shown below. Given that $h_1(n) = a^n u(n)$, $h_2(n) = \delta(n-1)$, $h_3(n) = \delta(n-2)$.



Solution

The given system can be reduced to single equivalent system as shown below.



Here, $h(n) = [h_1(n) * h_2(n)] + [h_3(n) * h_1(n)]$

Let us evaluate the convolution of $h_1(n)$ and $h_2(n)$.

$$h_1(n) * h_2(n) = \sum_{m=-\infty}^{\infty} h_1(m)\, h_2(n-m)$$

$$= \sum_{m=-\infty}^{\infty} h_2(m)\, h_1(n-m) \qquad \boxed{\text{Using commutative property}}$$

$$= \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^{(n-m)} = \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^n\, a^{-m}$$

$$= a^n \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^{-m}$$

The product of $\delta(m-1)$ and a^{-m} in the above equation will be non-zero only when $m = 1$.

$$\therefore\ h_1(n) * h_2(n) = a^n a^{-1} = a^{n-1} \quad ; \text{ for } n \geq 1$$

$$= a^{n-1} u(n-1) \quad ; \text{ for all n.}$$

Let us evaluate the convolution of $h_3(n)$ and $h_1(n)$.

$$h_3(n) * h_1(n) = \sum_{m=-\infty}^{\infty} h_3(m)\, h_1(n-m)$$

$$h_3(n) * h_1(n) = \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^{(n-m)} = \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^n\, a^{-m}$$

$$= a^n \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^{-m}$$

The product of $\delta(m-2)$ and a^{-m} in the above equation will be non-zero only when m = 2.

$$\therefore h_1(n) * h_2(n) = a^n a^{-2} = a^{n-2} \quad ; \quad \text{for } n \geq 2$$

$$= a^{n-2} u(n-2) \quad ; \quad \text{for all } n$$

Now, the overall impulse response h(n) is given by,

$$h(n) = [h_1(n) * h_2(n)] + [h_3(n) * h_1(n)]$$

$$= a^{(n-1)} u(n-1) + a^{(n-2)} u(n-2)$$

6.9.5 Methods of Performing Linear Convolution

Method -1: Graphical Method

Let $x_1(n)$ and $x_2(n)$ be the input sequences and $x_3(n)$ be the output sequence.

1. Change the index "n" of input sequences to "m" to get $x_1(m)$ and $x_2(m)$.
2. Sketch the graphical representation of the input sequences $x_1(m)$ and $x_2(m)$.
3. Let us fold $x_2(m)$ to get $x_2(-m)$. Sketch the graphical representation of the folded sequence $x_2(-m)$.
4. Shift the folded sequence $x_2(-m)$ to the left graphically so that the product of $x_1(m)$ and shifted $x_2(-m)$ gives only one non-zero sample. Now multiply $x_1(m)$ and shifted $x_2(-m)$ to get a product sequence, and then sum-up the samples of product sequence, which is the first sample of output sequence.
5. To get the next sample of output sequence, shift $x_2(-m)$ of previous step to one position right and multiply the shifted sequence with $x_1(m)$ to get a product sequence. Now the sum of the samples of product sequence gives the second sample of output sequence.
6. To get subsequent samples of output sequence, the step-5 is repeated until we get a non-zero product sequence.

Method -2: Tabular Method

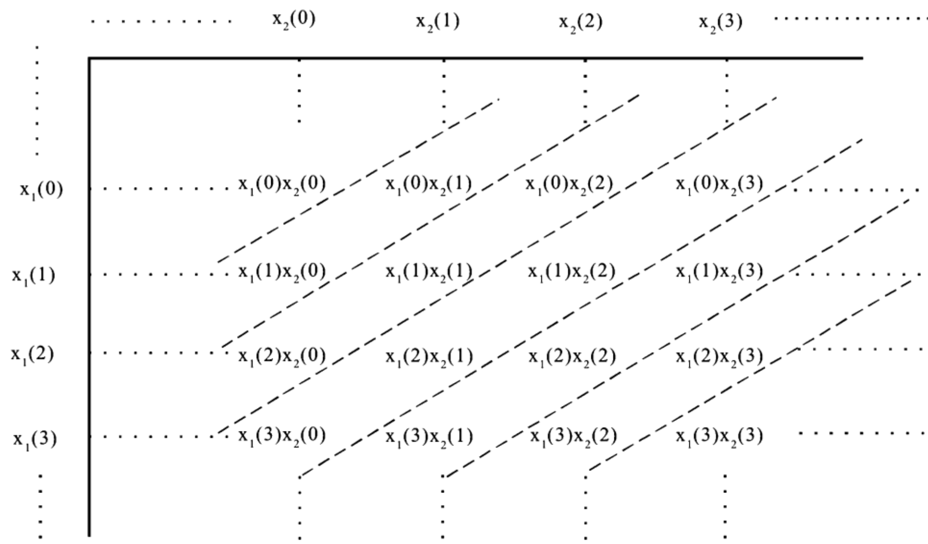
The tabular method is same as that of graphical method, except that the tabular representation of the sequences are employed instead of graphical representation. In tabular method, every input sequence, folded and shifted sequence is represented by a row in a table.

Method -3: Matrix Method

Let $x_1(n)$ and $x_2(n)$ be the input sequences and $x_3(n)$ be the output sequence. In matrix method one of the sequences is represented as a row and the other as a column as shown below.

Multiply each column element with row elements and fill up the matrix array.

Now the sum of the diagonal elements gives the samples of output sequence $x_3(n)$. (The sum of the diagonal elements are shown below for reference).



$$x_3(0) = ... + x_1(0)\ x_2(0) + ...$$

$$x_3(1) = ... + x_1(1)\ x_2(0) + x_1(0\ )\ x_2(1) + ...$$

$$x_3(2) = ... + x_1(2)\ x_2(0) + x_1(1)\ x_2(1) + x_1(0)\ x_2(2) + ...$$

$$x_3(3) = ... + x_1(3)\ x_2(0) + x_1(2)\ x_2(1) + x_1(1)\ x_2(2) + x_1(0)\ x_2(3) + ...$$

$$\vdots$$

$$\therefore\ x_3(n) = \{.....\ x_3(0),\ x_3(1),\ x_3(2),\ x_3(3),\ .....\}$$

Example 6.22

Determine the response of the LTI system whose input x(n) and impulse response h(n) are given by, $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$ and $h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$

Solution

The response y(n) of the system is given by convolution of x(n) and h(n).

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\ h(n-m)$$

In this example the convolution operation is performed by three methods.

The Input sequence starts at n = 0 and the impulse response sequence starts at n = –1. Therefore the output sequence starts at n = 0 + (–1) = –1.

The input and impulse response consists of 4 samples, so the output consists of 4 + 4 – 1 = 7 samples.

Method 1 : Graphical Method

The graphical representation of x(n) and h(n) after replacing n by m are shown below. The sequence h(m) is folded with respect to m = 0 to obtain h(–m).

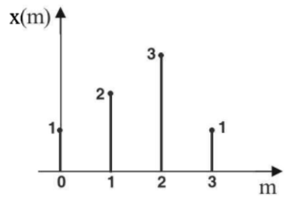


***Fig 1** : Input sequence.*

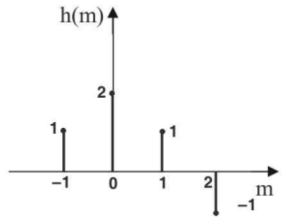


***Fig 2** : Impulse response.*

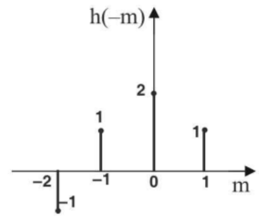


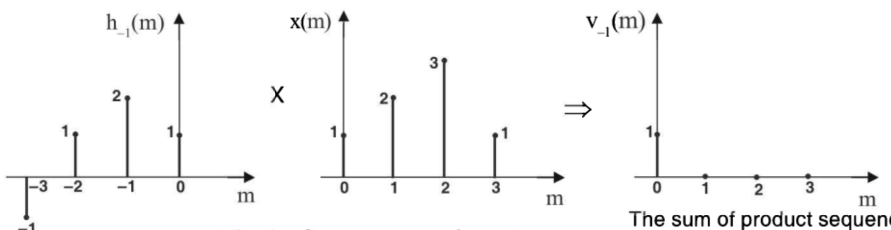
***Fig 3** : Folded impulse response.*

The samples of y(n) are computed using the convolution formula,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m)\ ;\ \text{where } h_n(m) = h(n-m)$$

The computation of each sample using the above equation are graphically shown in fig 4 to fig 10. The graphical representation of output sequence is shown in fig 11.

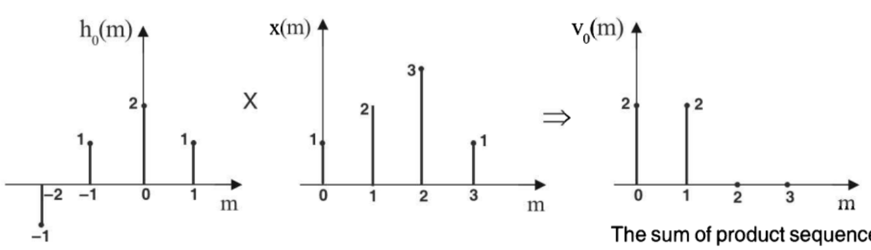
When n = –1 ; $y(-1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(-1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_{-1}(m) = \sum_{m=-\infty}^{+\infty} v_{-1}(m)$



The sum of product sequence $v_{-1}(m)$ gives y(–1). ∴ y(–1) = 1

***Fig 4 :** Computation of y(–1).*

When n = 0 ; $y(0) = \sum_{m=-\infty}^{+\infty} x(m)\, h(0-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_0(m) = \sum_{m=-\infty}^{+\infty} v_0(m)$



The sum of product sequence $v_0(m)$ gives y(0). ∴ y(0) = 2 + 2 = 4

***Fig 5 :** Computation of y(0).*

When $n = 1$; $y(1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_1(m) = \sum_{m=-\infty}^{+\infty} v_1(m)$

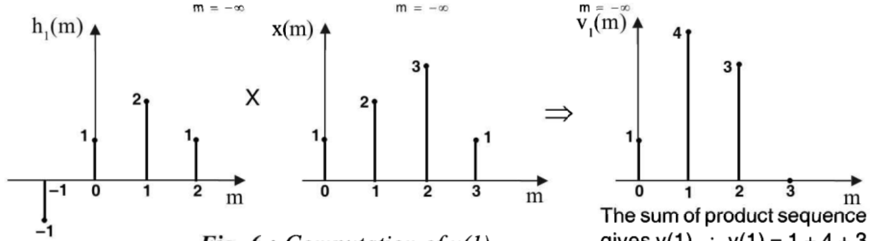


Fig 6 : Computation of y(1).

The sum of product sequence $v_1(m)$ gives y(1). $\therefore$ $y(1) = 1 + 4 + 3 = 8$

When $n = 2$; $y(2) = \sum_{m=-\infty}^{+\infty} x(m)\, h(2-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_2(m) = \sum_{m=-\infty}^{+\infty} v_2(m)$

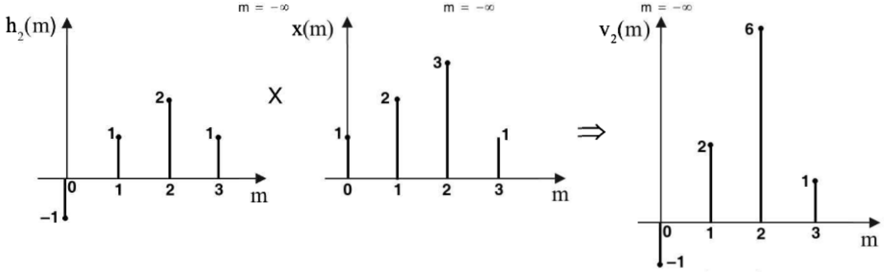


Fig 7 : Computation of y(2).

The sum of product sequence $v_2(m)$ gives y(2). $\therefore$ $y(2) = -1 + 2 + 6 + 1 = 8$

When $n = 3$; $y(3) = \sum_{m=-\infty}^{+\infty} x(m)\, h(3-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_3(m) = \sum_{m=-\infty}^{+\infty} v_3(m)$

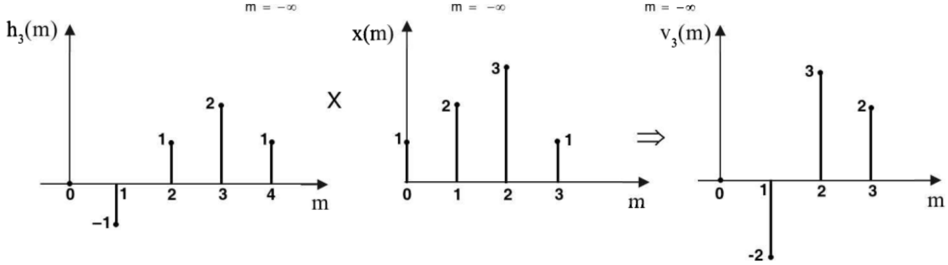


Fig 8 : Computation of y(3).

The sum of product sequence $v_3(m)$ gives y(3). $\therefore$ $y(3) = -2 + 3 + 2 = 3$

When $n = 4$; $y(4) = \sum_{m=-\infty}^{+\infty} x(m)\, h(4-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_4(m) = \sum_{m=-\infty}^{+\infty} v_4(m)$

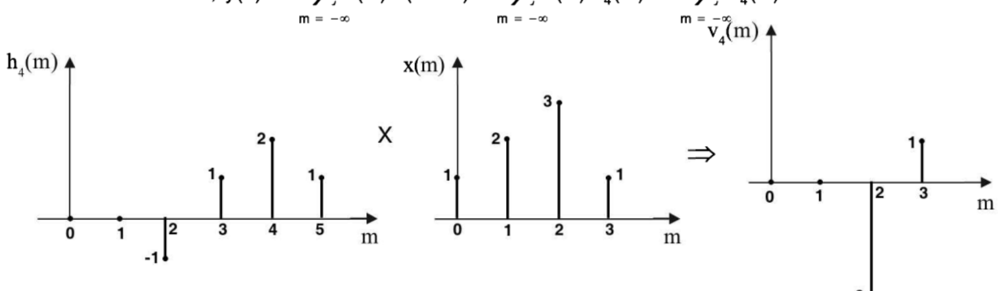
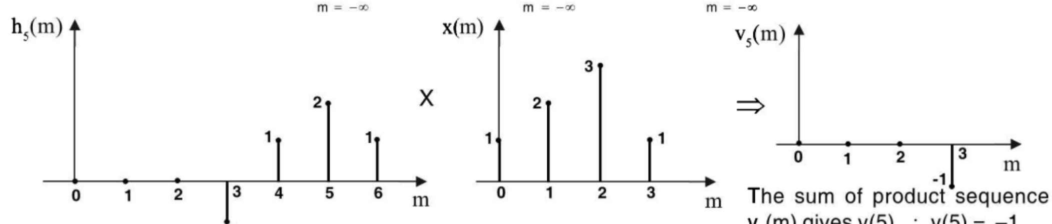


Fig 9 : Computation of y(4).

The sum of product sequence $v_4(m)$ gives y(4). $\therefore$ $y(4) = -3 + 1 = -2$

When n = 5 ; $y(5) = \sum_{m=-\infty}^{+\infty} x(m)\, h(5-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_5(m) = \sum_{m=-\infty}^{+\infty} v_5(m)$



The sum of product sequence $v_5(m)$ gives y(5). $\therefore$ y(5) = −1

Fig 10 : Computation of y(5).

The output sequence, $y(n) = \left\{1,\ \underset{\uparrow}{4},\ 8,\ 8,\ 3,\ -2,\ -1\right\}$

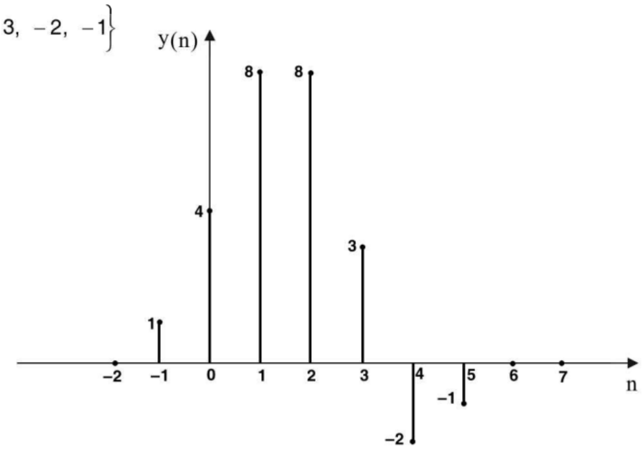


Fig 11 : Graphical representation of y(n).

Method - 2 : Tabular Method

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

***Note :** The unfilled boxes in the table are considered as zeros.*

m	–3	–2	–1	0	1	2	3	4	5	6
x(m)				1	2	3	1			
h(m)			1	2	1	–1				
h(–m)		–1	1	2	1					
$h(-1-m) = h_{-1}(m)$	–1	1	2	1						
$h(0-m) = h_0(m)$		–1	1	2	1					
$h(1-m) = h_1(m)$			–1	1	2	1				
$h(2-m) = h_2(m)$				–1	1	2	1			
$h(3-m) = h_3(m)$					–1	1	2	1		
$h(4-m) = h_4(m)$						–1	1	2	1	
$h(5-m) = h_5(m)$							–1	1	2	1

Each sample of y(n) is computed using the convolution formula,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m), \quad \text{where } h_n(m) = h(n-m)$$

To determine a sample of y(n) at n = q, multiply the sequence x(m) and $h_q(m)$ to get a product sequence (i.e., multiply the corresponding elements of the row x(m) and $h_q(m)$). The sum of all the samples of the product sequence gives y(q).

$$\text{When } n=-1\ ;\ y(-1) = \sum_{m=-3}^{3} x(m)\, h_{-1}(m) \qquad \boxed{\because \text{ The product is valid only for } m = -3 \text{ to } +3}$$

$$= x(-3)\, h_{-1}(-3) + x(-2) h_{-1}(-2) + x(-1) h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2)\, h_{-1}(2) + x(3)\, h_{-1}(3)$$

$$= 0 + 0 + 0 + 1 + 0 + 0 + 0 = 1$$

The samples of y(n) for other values of n are calculated as shown for n = –1.

$$\text{When } n = 0\ ;\ y(0) = \sum_{m=-2}^{3} x(m)\, h_0(m) = 0 + 0 + 2 + 2 + 0 + 0 = 4$$

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=-1}^{3} x(m)\, h_1(m) = 0 + 1 + 4 + 3 + 0 = 8$$

$$\text{When } n = 2\ ;\ y(2) = \sum_{m=0}^{3} x(m)\, h_2(m) = -1 + 2 + 6 + 1 = 8$$

$$\text{When } n = 3\ ;\ y(3) = \sum_{m=0}^{4} x(m)\, h_3(m) = 0 - 2 + 3 + 2 + 0 = 3$$

$$\text{When } n = 4\ ;\ y(4) = \sum_{m=0}^{5} x(m)\, h_4(m) = 0 + 0 - 3 + 1 + 0 + 0 = -2$$

$$\text{When } n = 5\ ;\ y(5) = \sum_{m=0}^{6} x(m)\, h_5(m) = 0 + 0 + 0 - 1 + 0 + 0 + 0 = -1$$

The output sequence, $y(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$

Method - 3 : Matrix Method

The input sequence x(n) is arranged as a column and the impulse response is arranged as a row as shown below. The elements of the two dimensional array are obtained by multiplying the corresponding row element with the column element. The sum of the diagonal elements gives the samples of y(n).

$$\begin{array}{c|cccc} {}_{x(n)\downarrow}\backslash^{h(n)\rightarrow} & 1 & 2 & 1 & -1 \\ \hline 1 & 1\times1 & 1\times2 & 1\times1 & 1\times(-1) \\ 2 & 2\times1 & 2\times2 & 2\times1 & 2\times(-1) \\ 3 & 3\times1 & 3\times2 & 3\times1 & 3\times(-1) \\ 1 & 1\times1 & 1\times2 & 1\times1 & 1\times(-1) \end{array} \Rightarrow \begin{array}{c|cccc} {}_{x(n)\downarrow}\backslash^{h(n)\rightarrow} & 1 & 2 & 1 & 1 \\ \hline 1 & 1 & 2 & 1 & -1 \\ 2 & 2 & 4 & 2 & -2 \\ 3 & 3 & 6 & 3 & -3 \\ 1 & 1 & 2 & 1 & -1 \end{array}$$

y(–1) = 1

y(0) = 2 + 2 = 4

y(1) = 3 + 4 + 1 = 8

y(2) = 1 + 6 + 2 + (–1) = 8

y(3) = 2 + 3 + (–2) = 3

y(4) = 1 + (–3) = –2

y(5) = –1

$\therefore y(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$

Example 6.23

Determine the output y(n) of a relaxed LTI system with impulse response,

$h(n) = a^n\, u(n)$; where $|a| < 1$ and

When input is a unit step sequence, i.e., $x(n) = u(n)$.

Solution

The graphical representation of x(n) and h(n) after replacing n by m are shown below. Also the sequence x(m) is folded to get x(–m)

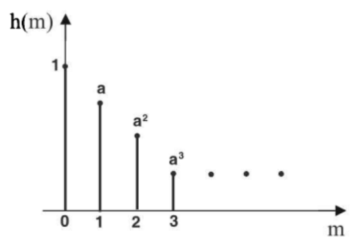


Fig 1 : Impulse response.

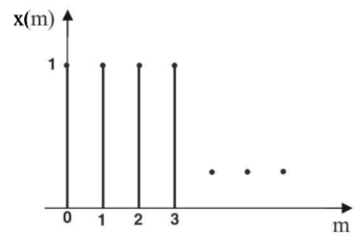


Fig 2 : Input sequence.

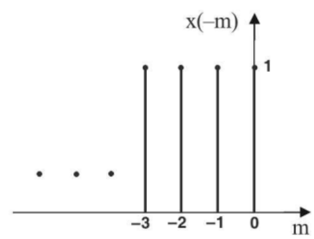


Fig 3 : Folded input sequence.

Here both h(m) and x(m) are infinite duration sequences starting at n = 0. Hence the output sequence y(n) will also be an infinite duration sequence starting at n = 0

By convolution formula,

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)\, x(n-m) = \sum_{m=0}^{\infty} h(m)\, x_n(m)\,; \quad \text{where } x_n(m) = x(n-m)$$

The computation of some samples of y(n) using the above equation are graphically shown below.

When n = 0 ; $y(0) = \sum_{m=0}^{\infty} h(m)\, x(m) = \sum_{m=0}^{\infty} h(m)\, x_0(m) = \sum_{m=0}^{\infty} v_0(m)$

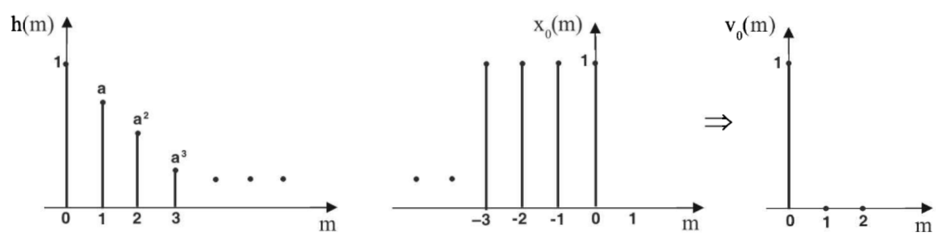


Fig 4 : Computation of y(0).

y(0) = 1

When n = 1 ; $y(1) = \sum_{m=0}^{\infty} h(m)\, x(1-m) = \sum_{m=0}^{\infty} h(m)\, x_1(m) = \sum_{m=0}^{\infty} v_1(m)$

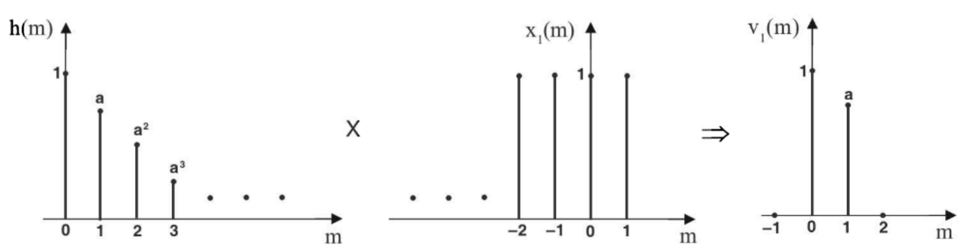


Fig 5 : Computation of y(1).

y(1) = 1 + a

When n = 2 ; $y(2) = \sum_{m=0}^{\infty} h(m)\,x(2-m) = \sum_{m=0}^{\infty} h(m)\,x_2(m) = \sum_{m=0}^{\infty} v_2(m)$

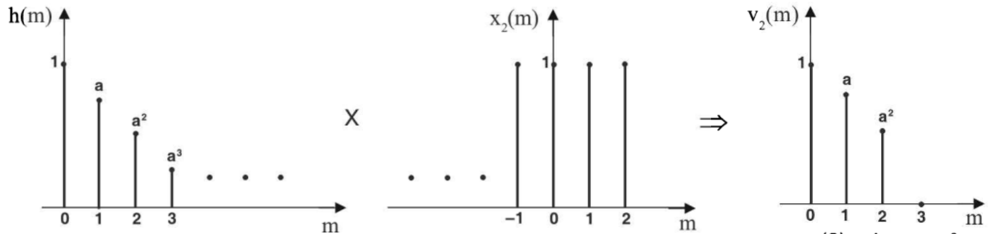


Fig 6 : Computation of y(2).

Solving similarly for other values of n, we can write y(n) for any value of n as shown below.

$$y(n) = 1 + a + a^2 + ...+ a^n = \sum_{p=0}^{n} a^p \quad ; \text{ for } n \geq 0$$

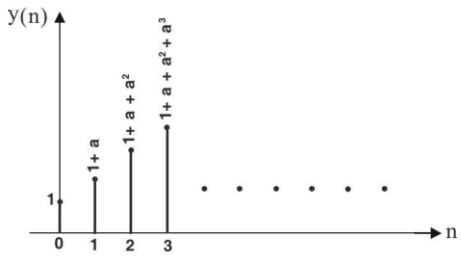


Fig 7 : Graphical representation of y(n).

6.10 Circular Convolution

6.10.1 Circular Representation and Circular Shift of Discrete Time Signal

Consider a finite duration sequence x(n) and its periodic extension $x_p(n)$. The periodic extension of x(n) can be expressed as $x_p(n) = x(n + N)$, where N is the periodicity. Let N = 4. The sequence x(n) and its periodic extension are shown in fig 6.24.

$$\text{Let, } x(n) = \begin{aligned} &1 \; ; \; n = 0 \\ &= 2 \; ; \; n = 1 \\ &= 3 \; ; \; n = 2 \\ &= 4 \; ; \; n = 3 \end{aligned}$$

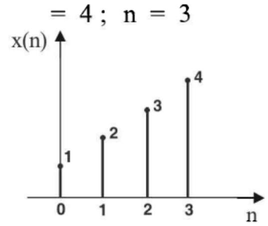


Fig 6.24a : Finite duration sequence x(n).

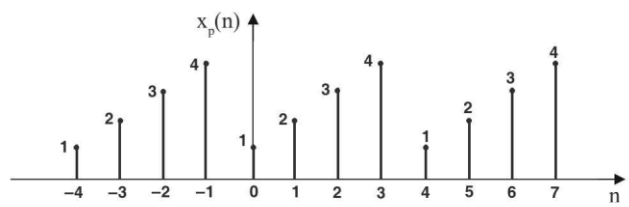


Fig 6.24b : Periodic extension of x(n).

Fig 6.24 : A finite duration sequence and its periodic extension.

Let us delay the periodic sequence $x_p(n)$ by two units of time as shown in fig 6.25(a). (For delay the sequence is shifted right). Let us denote one period of this delayed sequence by $x_1(n)$. One period of the delayed sequence is shown in figure 6.25(b).

UNIT-5

Structures for Realization of IIR and FIR Systems

10.1 Introduction

A discrete time system is a system that accepts a discrete time signal as input and processes it, and delivers the processed discrete time signal as output. Mathematically, a discrete time system is represented by a difference equation. Physically, a discrete time system is realized or implemented either as a digital hardware (like special purpose Microprocessor / Microcontroller) or as a software running on a digital hardware (like PC-Personal Computer).

The processing of the discrete time signal by the digital hardware involves mathematical operations like addition, multiplication, and delay. Also the calculations are performed either by using fixed point arithmetic or floating point arithmetic. The time taken to process the discrete time signal and the computational complexity, depends on number of calculations involved and the type of arithmetic used for computation. These issues are addressed in structures for realization of discrete time systems.

From the implementation point of view, the discrete time systems are basically classified as IIR and FIR systems. The various structures proposed for IIR and FIR systems, attempt to reduce the computational complexity, errors in computation and the memory requirement of the system.

10.2 Discrete Time IIR and FIR Systems

A discrete time system is usually designed for a specified frequency response, $H(e^{j\omega})$. Now, the impulse response, h(n) of the system is given by inverse Fourier transform of the frequency response, $H(e^{j\omega})$. The impulse response, h(n) will be a sequence with infinite samples.

When a discrete time system is designed by considering all the infinite samples of the impulse response, then the system is called ***IIR (Infinite Impulse Response) system***. When a discrete time system is designed by choosing only finite samples (usually N-samples) of the impulse response, then the system is called ***FIR (Finite Impulse Response) system***.

In the design of IIR systems, the infinite samples of impulse response cannot be handled in digital domain. Therefore, the frequency response of IIR system will be transferred to a corresponding frequency response of a continuous time system, and a continuous time system is designed, then the continuous time system is transformed to discrete time system.

10.2.1 Discrete Time IIR System

Let, $H(e^{j\omega})$ = Frequency response of discrete time IIR system

H(s) = Transfer function of continuous time system

H(z) = Transfer function of discrete time system

The frequency response of discrete time system is transferred to a corresponding frequency response of continuous time system. Using this frequency response of continuous time system the transfer function of continuous time system is designed. Then the transfer function of continuous time system, H(s) is transformed to transfer function of discrete time system, H(z).

The general form of H(z) is,

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}}$$

Let, X(z) = Input of the discrete time system in z-domain

Y(z) = Output of the discrete time system in z-domain

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}} \qquad .....(10.1)$$

On cross multiplying the equation (10.1) we get,

$$\left[1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}\right]Y(z) = \left[b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}\right]X(z)$$

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + ... + a_N z^{-N} Y(z)$$
$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + ... + b_M z^{-M} X(z)$$

On taking inverse $\mathcal{Z}$-transform of the above equation we get,

$\boxed{\text{If } \mathcal{Z}\{x(n)\} = X(z) \text{ then, } \mathcal{Z}\{x(n-k)\} = z^{-k} X(z)}$

$$y(n) + a_1 y(n-1) + a_2 y(n-2) + ... + a_N y(n-N)$$
$$= b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + ... + b_M x(n-M)$$

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - ..... - a_N y(n-N)$$
$$+ b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + ... + b_M x(n-M)$$

$$\therefore y(n) = -\sum_{m=1}^{N} a_m y(n-m) + \sum_{m=0}^{M} b_m x(n-m) \qquad .....(10.2)$$

The equation (10.1) is the transfer function of discrete time IIR system and the equation (10.2) is the time domain equation governing discrete time IIR system. From equation (10.2), it is observed that the output at any time n depends on past outputs and so the IIR systems are recursive systems.

10.2.2 Discrete Time FIR system

Let, $H(e^{j\omega})$ = Frequency response of discrete time FIR system

h(n) = Impulse response of discrete time FIR system

Here, the impulse response is obtained by inverse Fourier transform of the frequency response of discrete time system. The impulse response will have infinite samples. Let us choose N-samples of h(n) for n = 0 to N–1. (or for n = –(N–1)/2 to + (N–1)/2).

Let the samples of h(n) be, $b_0, b_1, b_2, ... b_{N-1}$ for n = 0, 1, 2, ... N–1 respectively.

$$\therefore h(n) = \{ \underset{\uparrow}{b_0}, b_1, b_2, ... b_{N-1} \}$$

On taking $\mathcal{Z}$-transform of h(n) we get,

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)}$$

Let, X(z) = Input of the discrete time system in z-domain

Y(z) = Output of the discrete time system in z-domain

$$\therefore H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)} \quad .....(10.3)$$

On cross multiplying the equation (10.3) we get,

$$Y(z) = \left[b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)}\right]X(z)$$
$$= b_0X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + ... + b_{N-1} z^{-(N-1)} X(z)$$

On taking inverse $\mathcal{Z}$-transform of the above equation we get,

$$y(n) = b_0x(n) + b_1 x(n-1) + b_2 x(n-2) + ... + b_{N-1} x(n-(N-1))$$

$$\therefore y(n) = \sum_{m=0}^{N-1} b_m x(n-m) \quad .....(10.4)$$

The equation (10.3) is the transfer function of discrete time FIR system and the equation (10.4) is the time domain equation governing discrete time FIR system. From equation (10.4), it is observed that the output at any time n does not depend on past outputs and so the FIR systems are nonrecursive systems.

10. 3 Structures for Realization of IIR Systems

In general, the time domain representation of an N^{th} order IIR system is,

$$y(n) = -\sum_{m=1}^{N} a_m y(n-m) + \sum_{m=0}^{M} b_m x(n-m)$$

and the z-domain representation of an N^{th} order IIR system is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}}$$

The above two representations of IIR system can be viewed as a computational procedure (or algorithm) to determine the output sequence y(n) from the input sequence x(n). Also, in the above representations the value of M gives the number of zeros and the value of N gives the number of poles of the IIR system.

The computations in the above equation can be arranged into various equivalent sets of difference equations, with each set of equations defining a computational procedure or algorithm for implementing the system. The main advantage of rearranging the sets of difference equations is to reduce the computational complexity, memory requirements and finite-word-length effects in computations.

For each set of equations, we can construct a block diagram consisting of delays, adders and multipliers. Such block diagrams are referred as realization of system or equivalently as a structure for realizing system. (For the block diagram representation of discrete system refer chapter-6, section 6.6.2). Some of the block diagram representation of the system gives a direct relation between the time domain equation and the z-domain equation.

The different types of structures for realizing the IIR systems are,

1. Direct form-I structure
2. Direct form-II structure
3. Cascade form structure
4. Parallel form structure

10.3.1 Direct Form-I Structure of IIR System

Consider the difference equation governing an IIR system.

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - \ldots\ldots - a_N\, y(n-N) + b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \ldots\ldots + b_M\, x(n-M)$$

On taking Z-transform of the above equation we get,

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \ldots\ldots - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \ldots\ldots + b_M z^{-M} X(z) \quad \ldots..(10.5)$$

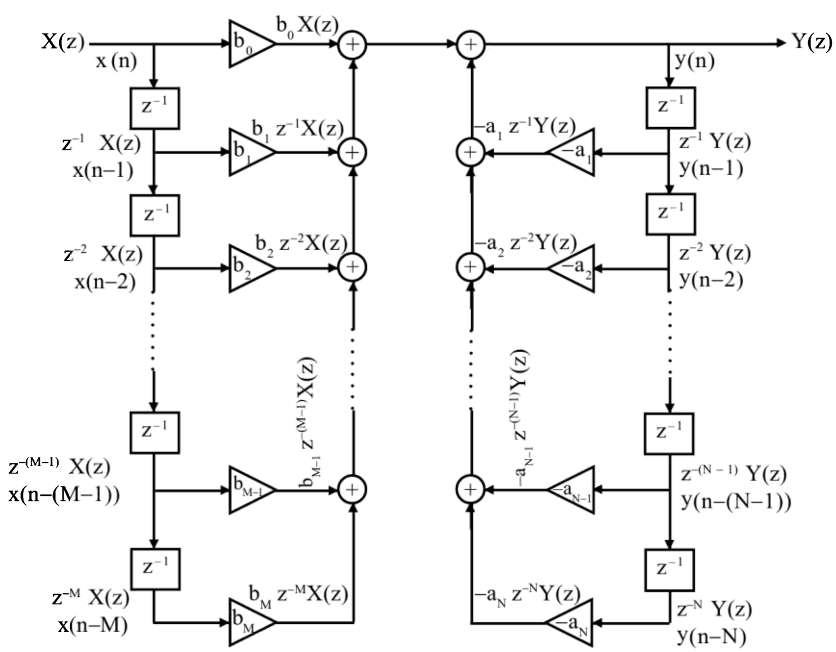


***Fig 10.1 :** Direct form-I structure of IIR system.*

The equation of Y(z) [equation (10.5)] can be directly represented by a block diagram as shown in fig 10.1 and this structure is called direct form-I structure. The direct form-I structure provides a direct relation between time domain and z-domain equations. The direct form-I structure uses separate delays (z^{-1}) for input and output samples. Hence for realizing direct form-I structure more memory is required.

From the direct form-I structure it is observed that the realization of an N^{th} order discrete time system with M number of zeros and N number of poles, involves M+N+1 number of multiplications and M+N number of additions. Also this structure involves M+N delays and so M+N memory locations are required to store the delayed signals.

When the number of delays in a structure is equal to the order of the system, the structure is called ***canonic structure***. In direct form-I structure the number of delays is not equal to order of the system and so direct form-I structure is noncanonic structure.

10.3.2 Direct Form-II Structure of IIR System

An alternative structure called direct form-II structure can be realized which uses less number of delay elements than the direct form-I structure.

Consider the general difference equation governing an IIR system.

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - \ldots\ldots - a_N\, y(n-N)$$
$$+\, b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \ldots\ldots + b_M\, x(n-M)$$

On taking Z-transform of the above equation we get,

$$Y(z) = -a_1 z^{-1}\, Y(z) - a_2 z^{-2}\, Y(z) - \ldots\ldots - a_N z^{-N}\, Y(z)$$
$$+\, b_0\, X(z) + b_1 z^{-1}\, X(z) + b_2 z^{-2}\, X(z) + \ldots\ldots + b_M z^{-M}\, X(z)$$

$$Y(z) + a_1 z^{-1}\, Y(z) + a_2 z^{-2}\, Y(z) + \ldots\ldots + a_N z^{-N}\, Y(z)$$
$$= b_0\, X(z) + b_1 z^{-1}\, X(z) + b_2 z^{-2}\, X(z) + \ldots\ldots + b_M z^{-M}\, X(z)$$

$$Y(z)\left[1 + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots + a_N z^{-N}\right]$$
$$= X(z)\left[b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots + b_M z^{-M}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots + a_N z^{-N}}$$

Let, $$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)}$$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots + a_N z^{-N}} \qquad \ldots\ldots(10.6)$$

$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots + b_M z^{-M} \qquad \ldots\ldots(10.7)$$

On cross multiplying equation (10.6) we get,

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) + ... + a_N z^{-N} W(z) = X(z)$$

$$\therefore W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - ..... - a_N z^{-N} W(z) \quad ..... (10.8)$$

On cross multiplying equation (10.7) we get,

$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + ... + b_M z^{-M} W(z) \quad ..... (10.9)$$

The equations (10.8) and (10.9) represent the IIR system in z-domain and can be realized by a direct structure called direct form-II structure as shown in fig 10.2. In direct form-II structure the number of delays is equal to order of the system and so the direct form-II structure is canonic structure.

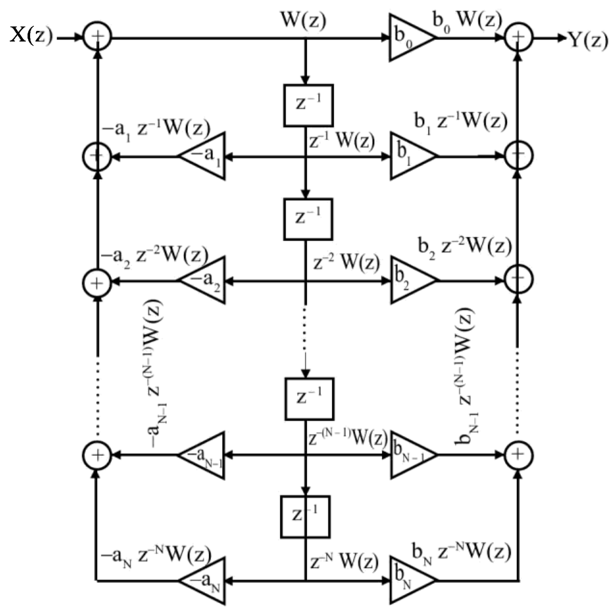


Fig 10.2 : *Direct form-II structure of IIR system for N = M.*

From the direct form-II structure it is observed that the realization of an N^{th} order discrete time system with M number of zeros and N number of poles, involves M+N+1 number of multiplications and M+N number of additions. In a realizable system, $N \geq M$, and so the number of delays in direct form-II structure will be equal to N. Hence, when a system is realized using direct form-II structure, N memory locations are required to store the delayed signals.

Conversion of Direct Form-I Structure to Direct Form-II Structure

The direct form-I structure can be converted to direct form-II structure by considering the direct form-I structure as cascade of two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ as shown in fig 10.3. By linearity property the order of cascading can be interchanged as shown in fig 10.4 and fig 10.5.

In fig 10.5 we can observe that the input to the delay elements in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same and so the output of delay elements in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same. Therefore instead of having separate delays for $\mathcal{H}_1$ and $\mathcal{H}_2$, a single set of delays can be used. Hence the delays can be merged to combine the cascaded systems to a single system and the resultant structure will be direct form-II structure as that of fig 10.2.

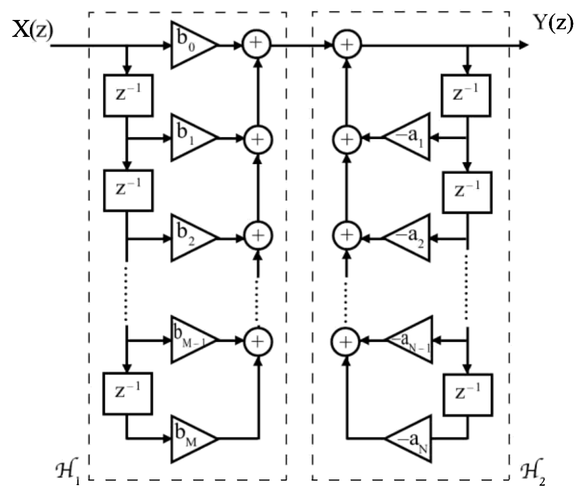


***Fig 10.3 :** Direct form-I structure as cascade of two systems.*

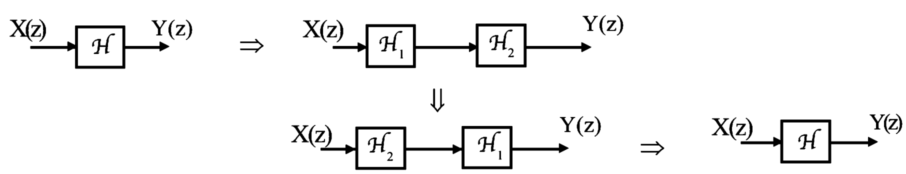


***Fig 10.4 :** Conversion of Direct form-I structure to Direct form-II structure.*

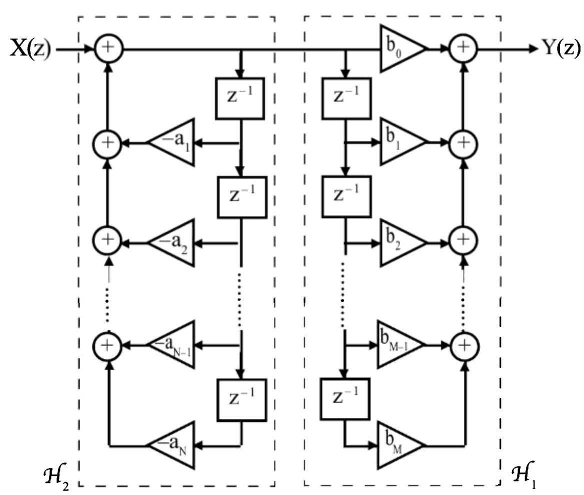


***Fig 10.5 :** Direct form-I structure after interchanging the order of cascading.*

10.3.3 Cascade Form Realization of IIR System

The transfer function H(z) can be expressed as a product of a number of second order or first order sections, as shown in equation (10.10).

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) \times H_2(z) \times H_3(z) \ldots\ldots H_m(z) = \prod_{i=1}^{m} H_i(z) \qquad \ldots\ldots(10.10)$$

$$\text{where, } H_i(z) = \frac{c_{0i} + c_{1i}\, z^{-1} + c_{2i}\, z^{-2}}{d_{0i} + d_{1i}\, z^{-1} + d_{2i}\, z^{-2}} \qquad \boxed{\text{Second order section}}$$

$$\text{or, } \quad H_i(z) = \frac{c_{0i} + c_{1i}\, z^{-1}}{d_{0i} + d_{1i}\, z^{-1}} \qquad \boxed{\text{First order section}}$$

The individual second order or first order sections can be realized either in direct form-I or direct form-II structures. The overall system is obtained by cascading the individual sections as shown in fig 10.6. The number of calculations and the memory requirement depends on the realization of individual sections.

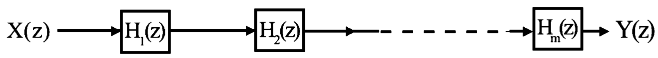


***Fig 10.6 :** Cascade form realization of IIR system.*

The difficulty in cascade structure are,

1. Decision of pairing poles and zeros.
2. Deciding the order of cascading the first and second order sections.
3. Scaling multipliers should be provided between individual sections to prevent the system variables from becoming too large or too small.

10.3.4 Parallel Form Realization of IIR System

The transfer function H(z) of a discrete time system can be expressed as a sum of first and second order sections, using partial fraction expansion technique as shown in equation (10.11).

$$H(z) = \frac{Y(z)}{X(z)} = C + H_1(z) + H_2(z) + \ldots\ldots + H_m(z) \qquad \ldots\ldots(10.11)$$

$$= C + \sum_{i=1}^{m} H_i(z)$$

$$\text{where, } H_i(z) = \frac{c_{0i} + c_{1i}\, z^{-1}}{d_{0i} + d_{1i}\, z^{-1} + d_{2i}\, z^{-2}} \qquad \boxed{\text{Second order section}}$$

$$\text{or } \quad H_i(z) = \frac{c_{0i}}{d_{0i} + d_{1i}\, z^{-1}} \qquad \boxed{\text{First order section}}$$

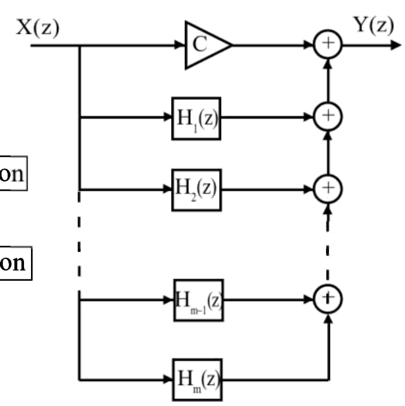


***Fig 10.7 :** Parallel form realization of IIR system.*

The individual first and second order sections can be realized either in direct form-I or direct form-II structures. The overall system is obtained by connecting the individual sections in parallel as shown in fig 10.7.The number of calculations and the memory requirement depends on the realization of individual sections.

Example 10.1

Obtain the direct form-I, direct form-II, cascade and parallel form realizations of the LTI system governed by the equation,

$$y(n) = -\frac{3}{8}\,y(n-1) + \frac{3}{32}\,y(n-2) + \frac{1}{64}\,y(n-3) + x(n) + 3\,x(n-1) + 2\,x(n-2).$$

Solution

Direct Form-I

Given that,

$$y(n) = -\frac{3}{8}\,y(n-1) + \frac{3}{32}\,y(n-2) + \frac{1}{64}\,y(n-3) + x(n) + 3\,x(n-1) + 2\,x(n-2) \qquad .....(1)$$

On taking Z-transform of equation(1) we get,

$$Y(z) = -\frac{3}{8}\,z^{-1}\,Y(z) + \frac{3}{32}\,z^{-2}\,Y(z) + \frac{1}{64}\,z^{-3}\,Y(z) + X(z) + 3z^{-1}\,X(z) + 2z^{-2}\,X(z) \qquad .....(2)$$

The direct form-I structure can be obtained from equation (2), as shown in fig 1.

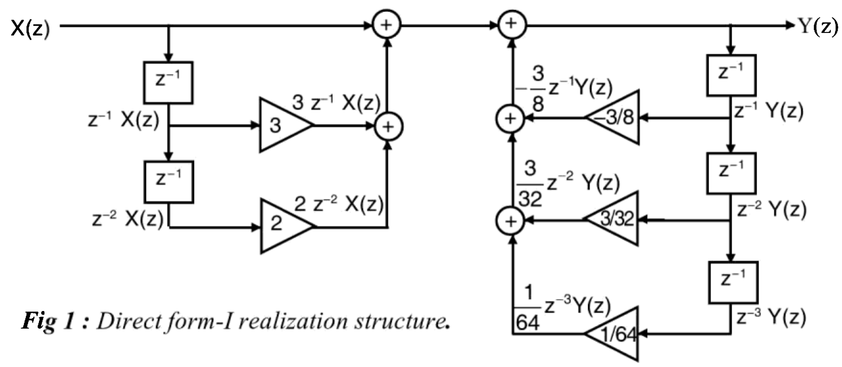


Fig 1 : Direct form-I realization structure.

Direct Form-II

Consider equation (2).

$$Y(z) = -\frac{3}{8}z^{-1}\,Y(z) + \frac{3}{32}z^{-2}\,Y(z) + \frac{1}{64}z^{-3}\,Y(z) + X(z) + 3z^{-1}\,X(z) + 2z^{-2}\,X(z)$$

$$Y(z) + \frac{3}{8}z^{-1}\,Y(z) - \frac{3}{32}z^{-2}\,Y(z) - \frac{1}{64}z^{-3}\,Y(z) = X(z) + 3z^{-1}\,X(z) + 2z^{-2}\,X(z)$$

$$Y(z)\left[1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}\right] = X(z)\left[1 + 3z^{-1} + 2z^{-2}\right]$$

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{1 + 3z^{-1} + 2z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \qquad .....(3)$$

Let, $\dfrac{Y(z)}{X(z)} = \dfrac{W(z)}{X(z)}\,\dfrac{Y(z)}{W(z)}$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \qquad .....(4)$$

$$\frac{Y(z)}{W(z)} = 1 + 3z^{-1} + 2z^{-2} \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$W(z)\left(1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}\right) = X(z)$$

$$\text{or}\quad W(z) = X(z) - \frac{3}{8}z^{-1}\,W(z) + \frac{3}{32}z^{-2}\,W(z) + \frac{1}{64}z^{-3}\,W(z) \qquad .....(6)$$

On cross multiplying equation (5) we get,

$$Y(z) = W(z) + 3z^{-1}\,W(z) + 2z^{-2}\,W(z) \qquad .....(7)$$

The equations (6) and (7) can be realized by a direct form-II structure as shown in fig 2.

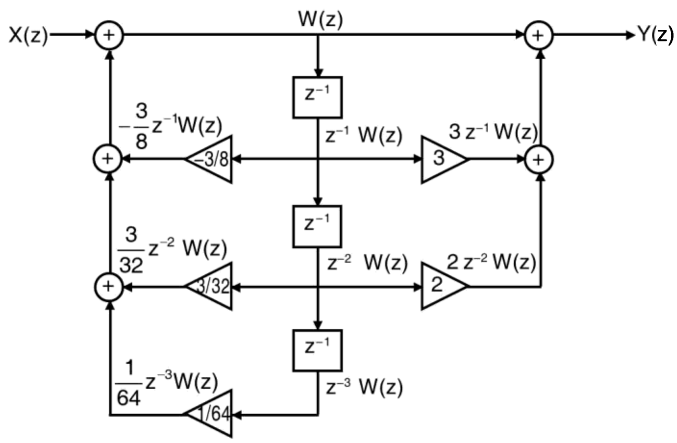


***Fig 2:** Direct form-II realization structure.*

Cascade Form

Consider equation (3).

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1 + 3z^{-1} + 2z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \qquad .....(8)$$

The numerator and denominator polynomials should be expressed in the factored form.

Consider the numerator polynomial of equation (8).

$$1 + 3z^{-1} + 2z^{-2} = z^{-2}\left(\frac{1}{z^{-2}} + \frac{3}{z^{-1}} + 2\right) = \frac{1}{z^2}\,(z^2 + 3z + 2)$$

$$= \frac{1}{z^2}\,(z + 1)\,(z + 2) = \frac{(z + 1)}{z}\;\frac{(z + 2)}{z}$$

$$= \left(1 + z^{-1}\right)\left(1 + 2z^{-1}\right) \qquad .....(9)$$

Consider the denominator polynomial of equation (8)

$$1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3} = z^{-3}\left(\frac{1}{z^{-3}} + \frac{3}{8}\,\frac{1}{z^{-2}} - \frac{3}{32}\,\frac{1}{z^{-1}} - \frac{1}{64}\right)$$

$$= \frac{1}{z^3}\left(z^3 + \frac{3}{8}z^2 - \frac{3}{32}z - \frac{1}{64}\right) \qquad .....(10)$$

$$= \frac{1}{z^3}\left(z + \frac{1}{8}\right)\left(z^2 + \frac{2}{8}z - \frac{8}{64}\right)$$

−1/8	1	3/8	−3/32	−1/64
	↓	−1/8	−2/64	+1/64
	1	2/8	−8/64	0

z = −1/8 is one of the root of the equation (10).

$$\therefore 1+\frac{3}{8}z^{-1}-\frac{3}{32}z^{-2}-\frac{1}{64}z^{-3}=\frac{1}{z^3}\left(z+\frac{1}{8}\right)\left(z^2+\frac{1}{4}z-\frac{1}{8}\right)$$

$$=\frac{1}{z^3}\left(z+\frac{1}{8}\right)\left(z+\frac{1}{2}\right)\left(z-\frac{1}{4}\right)$$

$$=\frac{\left(z+\frac{1}{8}\right)}{z}\;\frac{\left(z+\frac{1}{2}\right)}{z}\;\frac{\left(z-\frac{1}{4}\right)}{z}$$

$$=\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right) \qquad .....(11)$$

From equations(8), (9) and (11) we can write,

$$H(z)=\frac{1+3z^{-1}+2z^{-2}}{1+\frac{3}{8}z^{-1}-\frac{3}{32}z^{-2}-\frac{1}{64}z^{-3}}=\frac{(1+z^{-1})(1+2z^{-1})}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} \qquad .....(12)$$

Since there are three first order factors in the denominator of equation (12), H(z) can be expressed as a product of 3 sections as shown in equation (13).

$$\text{Let,}\quad H(z)=\frac{1+z^{-1}}{1+\frac{1}{8}z^{-1}}\times\frac{1+2z^{-1}}{1+\frac{1}{2}z^{-1}}\times\frac{1}{1-\frac{1}{4}z^{-1}}=H_1(z)\times H_2(z)\times H_3(z) \qquad .....(13)$$

$$\text{where,}\quad H_1(z)=\frac{1+z^{-1}}{1+\frac{1}{8}z^{-1}}\;;\quad H_2(z)=\frac{1+2z^{-1}}{1+\frac{1}{2}z^{-1}}\quad\text{and}\quad H_3(z)=\frac{1}{1-\frac{1}{4}z^{-1}}$$

The transfer function $H_1(z)$ can be realized in direct form-II structure as shown in fig 3.

$$\text{Let,}\quad H_1(z)=\frac{Y_1(z)}{X(z)}=\frac{W_1(z)}{X(z)}\,\frac{Y_1(z)}{W_1(z)}=\frac{1+z^{-1}}{1+\frac{1}{8}z^{-1}}$$

$$\text{where,}\quad \frac{W_1(z)}{X(z)}=\frac{1}{1+\frac{1}{8}z^{-1}}\quad\text{and}\quad\frac{Y_1(z)}{W_1(z)}=1+z^{-1}$$

$$\therefore\; W_1(z)=X(z)-\frac{1}{8}z^{-1}W_1(z)$$

$$Y_1(z)=W_1(z)+z^{-1}W_1(z)$$

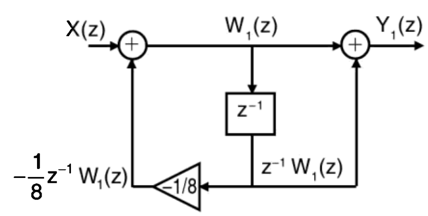


Fig 3 : Direct form-II structure of $H_1(z)$.

The transfer function $H_2(z)$ can be realized in direct form-II structure as shown in fig 4.

$$\text{Let,}\quad H_2(z)=\frac{Y_2(z)}{Y_1(z)}=\frac{W_2(z)}{Y_1(z)}\,\frac{Y_2(z)}{W_2(z)}=\frac{1+2z^{-1}}{1+\frac{1}{2}z^{-1}}$$

$$\text{where,}\quad \frac{W_2(z)}{Y_1(z)}=\frac{1}{1+\frac{1}{2}z^{-1}}\quad\text{and}\quad\frac{Y_2(z)}{W_2(z)}=1+2z^{-1}$$

$$\therefore\; W_2(z)=Y_1(z)-\frac{1}{2}z^{-1}W_2(z)$$

$$Y_2(z)=W_2(z)+2z^{-1}W_2(z)$$

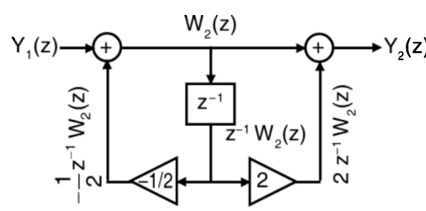


Fig 4 : Direct form-II structure of $H_2(z)$.

The transfer function $H_3(z)$ can be realized in direct form-II structure as shown in fig 5.

$$\text{Let,}\quad H_3(z)=\frac{Y(z)}{Y_2(z)}=\frac{1}{1-\frac{1}{4}z^{-1}}$$

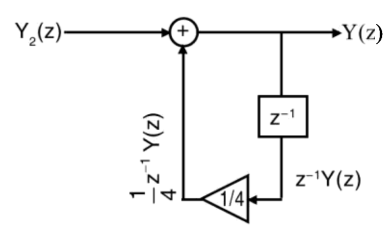


Fig 5 : Direct form-II structure of $H_3(z)$.

$$\therefore\ Y(z) - \frac{1}{4}z^{-1}\,Y(z) = Y_2(z)$$

$$Y(z) = Y_2(z) + \frac{1}{4}z^{-1}\,Y(z)$$

The cascade structure of the given system is obtained by connecting the individual sections shown in fig 3, fig 4 and fig 5 in cascade as shown in fig 6.

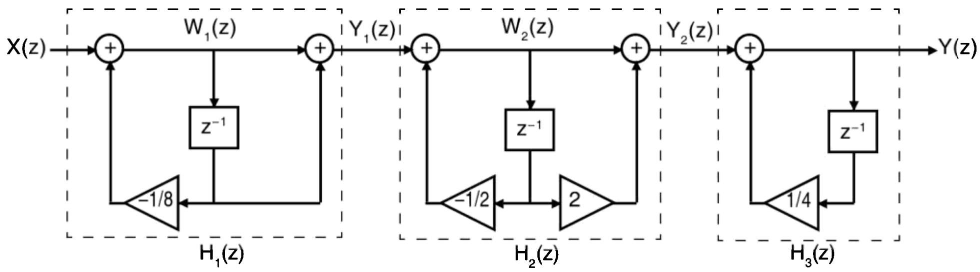


Fig 6 : *Cascade realization of the system.*

Parallel Form

Consider the equation (12).

$$H(z) = \frac{\left(1 + z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1 + \frac{1}{8}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}$$

By partial fraction expansion,

$$H(z) = \frac{A}{1 + \frac{1}{8}z^{-1}} + \frac{B}{1 + \frac{1}{2}z^{-1}} + \frac{C}{1 - \frac{1}{4}z^{-1}}$$

$$A = \left.\frac{\left(1 + z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1 + \frac{1}{8}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)} \times \left(1 + \frac{1}{8}z^{-1}\right)\right|_{z^{-1} = -8} = \frac{(1 - 8)\,(1 - 16)}{(1 - 4)\,(1 + 2)} = -\frac{35}{3}$$

$$B = \left.\frac{\left(1 + z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} \times \left(1+\frac{1}{2}z^{-1}\right)\right|_{z^{-1} = -2} = \frac{(1-2)\,(1-4)}{\left(1-\frac{1}{4}\right)\left(1+\frac{1}{2}\right)} = \frac{(-1)\times(-3)}{\frac{3}{4}\times\frac{3}{2}} = \frac{8}{3}$$

$$C = \left.\frac{\left(1 + z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} \times \left(1-\frac{1}{4}z^{-1}\right)\right|_{z^{-1} = 4} = \frac{(1+4)\,(1+8)}{\left(1+\frac{1}{2}\right)(1+2)} = \frac{5\times 9}{\frac{3}{2}\times 3} = 10$$

$$\therefore\ H(z) = \frac{-\frac{35}{3}}{1 + \frac{1}{8}z^{-1}} + \frac{\frac{8}{3}}{1 + \frac{1}{2}z^{-1}} + \frac{10}{1 - \frac{1}{4}z^{-1}} = H_1(z) + H_2(z) + H_3(z)$$

where, $H_1(z) = \dfrac{-\frac{35}{3}}{1 + \frac{1}{8}z^{-1}}$; $\qquad H_2(z) = \dfrac{\frac{8}{3}}{1 + \frac{1}{2}z^{-1}}$; $\qquad H_3(z) = \dfrac{10}{1 - \frac{1}{4}z^{-1}}$

Let, $H(z) = \frac{Y(z)}{X(z)}$; $H_1(z) = \frac{Y_1(z)}{X(z)}$; $H_2(z) = \frac{Y_2(z)}{X(z)}$; $H_3(z) = \frac{Y_3(z)}{X(z)}$

$$\therefore H(z) = H_1(z) + H_2(z) + H_3(z) \quad \Rightarrow \quad \frac{Y(z)}{X(z)} = \frac{Y_1(z)}{X(z)} + \frac{Y_2(z)}{X(z)} + \frac{Y_3(z)}{X(z)}$$

$$\therefore Y(z) = Y_1(z) + Y_2(z) + Y_3(z)$$

The transfer function $H_1(z)$ can be realized in direct form-I structure as shown in fig 7.

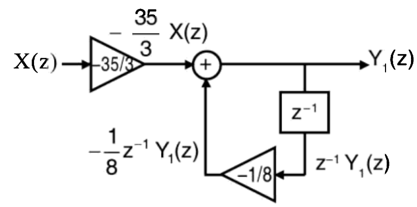


Fig 7 : Direct form-I structure of $H_1(z)$.

Let, $H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{-\frac{35}{3}}{1 + \frac{1}{8}z^{-1}}$

On cross multiplying and rearranging we get,

$$Y_1(z) = -\frac{1}{8}z^{-1}Y_1(z) - \frac{35}{3}X(z)$$

The transfer function $H_2(z)$ can be realized in direct form-I structure as shown in fig 8.

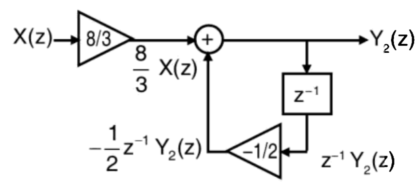


Fig 8 : Direct form-I structure of $H_2(z)$.

Let, $H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{\frac{8}{3}}{1 + \frac{1}{2}z^{-1}}$

On cross multiplying and rearranging we get,

$$Y_2(z) = -\frac{1}{2}z^{-1}Y_2(z) + \frac{8}{3}X(z)$$

The transfer function $H_3(z)$ can be realized in direct form-I structure as shown in fig 9.

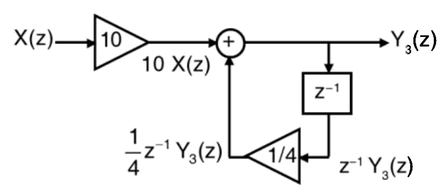


Fig 9 : Direct form-I structure of $H_3(z)$.

Let, $H_3(z) = \frac{Y_3(z)}{X(z)} = \frac{10}{1 - \frac{1}{4}z^{-1}}$

On cross multiplying and rearranging we get,

$$Y_3(z) = \frac{1}{4}z^{-1}Y_3(z) + 10\,X(z)$$

The overall structure is obtained by connecting the individual sections shown in fig 7, fig 8 and fig 9 in parallel as shown in fig 10.

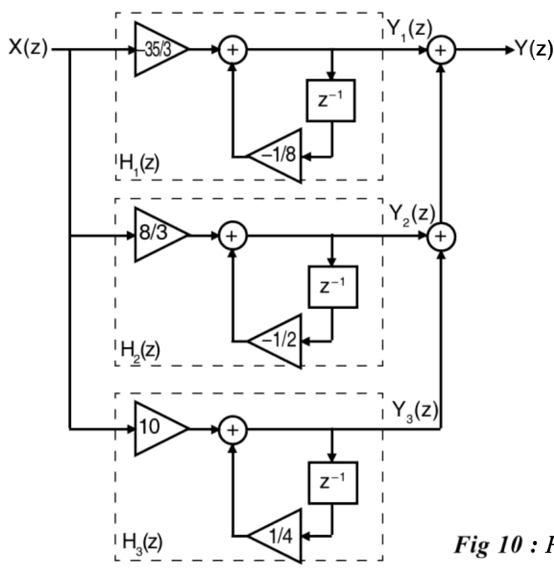


Fig 10 : Parallel form realization.

Example 10.2

Find the direct form-I and direct form-II realizations of a discrete time system represented by transfer function,

$$H(z) = \frac{8z^3 - 4z^2 + 11z - 2}{\left(z - \frac{1}{4}\right)\left(z^2 - z + \frac{1}{2}\right)}$$

Solution

Direct Form-I

Let, $H(z) = \frac{Y(z)}{X(z)}$; where, Y(z) = Output and X(z) = Input.

$$\therefore \frac{Y(z)}{X(z)} = \frac{8z^3 - 4z^2 + 11z - 2}{\left(z - \frac{1}{4}\right)\left(z^2 - z + \frac{1}{2}\right)} = \frac{8z^3 - 4z^2 + 11z - 2}{z^3 - z^2 + \frac{1}{2}z - \frac{1}{4}z^2 + \frac{1}{4}z - \frac{1}{8}}$$

$$= \frac{8z^3 - 4z^2 + 11z - 2}{z^3 - \frac{5}{4}z^2 + \frac{3}{4}z - \frac{1}{8}} = \frac{z^3\left(8 - 4z^{-1} + 11z^{-2} - 2z^{-3}\right)}{z^3\left(1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}\right)}$$

$$= \frac{8 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}} \qquad .....(1)$$

On cross multiplying equation (1) we get,

$$Y(z) - \frac{5}{4}z^{-1}Y(z) + \frac{3}{4}z^{-2}Y(z) - \frac{1}{8}z^{-3}Y(z) = 8X(z) - 4z^{-1}X(z) + 11z^{-2}X(z) - 2z^{-3}X(z)$$

$$\therefore Y(z) = 8X(z) - 4z^{-1}X(z) + 11z^{-2}X(z) - 2z^{-3}X(z)$$
$$+ \frac{5}{4}z^{-1}Y(z) - \frac{3}{4}z^{-2}Y(z) + \frac{1}{8}z^{-3}Y(z) \qquad .....(2)$$

The direct form-I structure can be obtained from equation (2) as shown in fig 1.

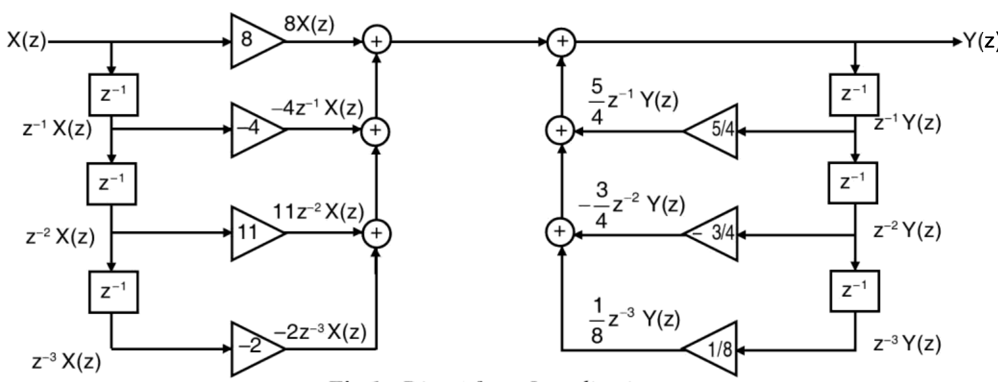


Fig 1 : Direct form-I realization.

Direct Form-II

From equation (1) we get,

$$\frac{Y(z)}{X(z)} = \frac{8 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}}$$

Let, $\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)}\,\frac{Y(z)}{W(z)}$

where, $\frac{W(z)}{X(z)} = \frac{1}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}}$ .....(3)

$$\frac{Y(z)}{W(z)} = 8 - 4z^{-1} + 11z^{-2} - 2z^{-3}$$ .....(4)

On cross multiplying equation (3) we get,

$$W(z) - \frac{5}{4}z^{-1}W(z) + \frac{3}{4}z^{-2}W(z) - \frac{1}{8}z^{-3}W(z) = X(z)$$

$$\therefore W(z) = X(z) + \frac{5}{4}z^{-1}W(z) - \frac{3}{4}z^{-2}W(z) + \frac{1}{8}z^{-3}W(z)$$ .....(5)

On cross multiplying equation (4) we get,

$$Y(z) = 8W(z) - 4z^{-1}W(z) + 11z^{-2}W(z) - 2z^{-3}W(z)$$ .....(6)

The equations (5) and (6) can be realized by a direct form-II Structure as shown in fig 2.

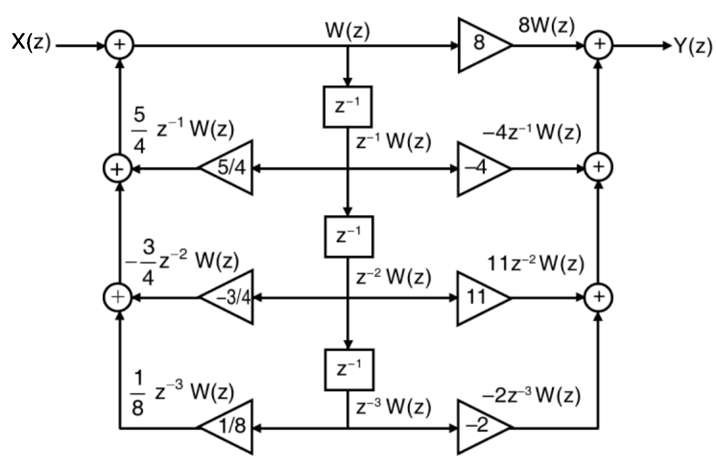


Fig 2 : Direct form-II realization.

Example 10.3

Find the digital network in direct form-I and II for the system described by the difference equation,

$$y(n) = x(n) + 0.5\,x(n-1) + 0.4\,x(n-2) - 0.6\,y(n-1) - 0.7\,y(n-2).$$

Solution

Given that, $y(n) = x(n) + 0.5\,x(n-1) + 0.4\,x(n-2) - 0.6\,y(n-1) - 0.7\,y(n-2)$

On taking $\mathcal{Z}$-transform we get,

$$Y(z) = X(z) + 0.5z^{-1}X(z) + 0.4z^{-2}X(z) - 0.6z^{-1}Y(z) - 0.7z^{-2}\,Y(z)$$ .....(1)

The direct form-I digital network can be realized using equation (1) as shown in fig 1.

On rearranging equation (1) we get,

$$Y(z) + 0.6z^{-1}Y(z) + 0.7z^{-2}Y(z) = X(z) + 0.5z^{-1}X(z) + 0.4z^{-2}X(z)$$

$$(1+0.6z^{-1} + 0.7z^{-2})\,Y(z) = (1 + 0.5z^{-1} + 0.4z^{-2})\,X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1 + 0.5z^{-1} + 0.4z^{-2}}{1 + 0.6z^{-1} + 0.7z^{-2}}$$ .....(2)

The equation (2) is the transfer function of the system.

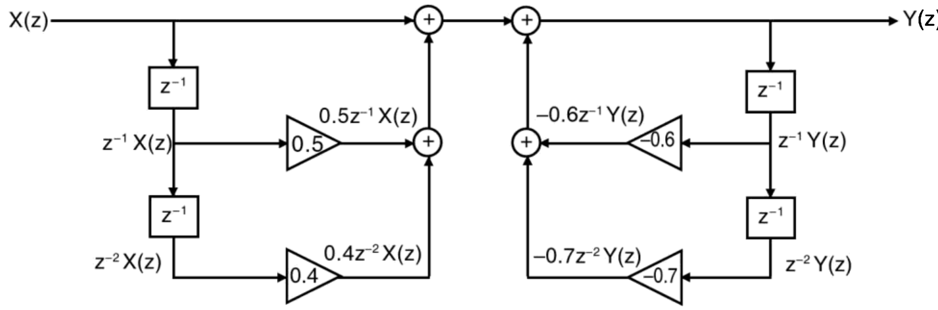


***Fig 1 :** Direct form-I digital network.*

Let, $\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \frac{Y(z)}{W(z)}$

where, $\frac{W(z)}{X(z)} = \frac{1}{1 + 0.6z^{-1} + 0.7z^{-2}}$ ..... (3)

$\frac{Y(z)}{W(z)} = 1 + 0.5z^{-1} + 0.4z^{-2}$ ..... (4)

On cross multiplying equation (3) we get,

$W(z) + 0.6z^{-1}W(z) + 0.7z^{-2}W(z) = X(z)$

$\therefore$ $W(z) = X(z) - 0.6z^{-1}W(z) - 0.7z^{-2}W(z)$..... (5)

On cross multiplying equation (4) we get,

$Y(z) = W(z) + 0.5z^{-1}W(z) + 0.4z^{-2}W(z)$ ..... (6)

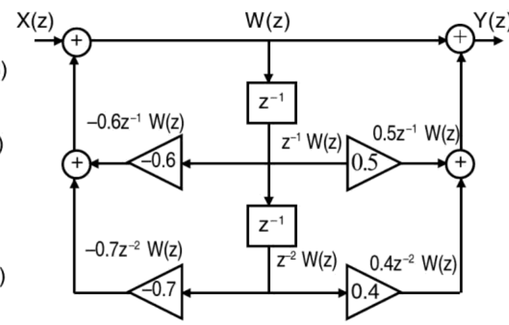


***Fig 2 :** Direct form-II digital network.*

The direct form-II digital network is realized using equations (5) and (6) as shown in fig 2.

Example 10.4

Realize the digital network described by H(z) in two ways. $H(z) = \frac{1 - r\cos\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$

Solution

Let, $H(z) = \frac{Y(z)}{X(z)} = \frac{1 - r\cos\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$

On cross multiplying we get,

$Y(z) - 2r\cos\omega_0 z^{-1}Y(z) + r^2 z^{-2}Y(z) = X(z) - r\cos\omega_0 z^{-1}X(z)$

$\therefore$ $Y(z) = X(z) - r\cos\omega_0 z^{-1}X(z) + 2r\cos\omega_0 z^{-1}Y(z) - r^2 z^{-2}Y(z)$

Let, $r\cos\omega_0 = a$. $\therefore$ $Y(z) = X(z) - az^{-1}X(z) + 2az^{-1}Y(z) - r^2 z^{-2}Y(z)$ ..... (1)

The equation (1) can be used to construct direct form-I structure of H(z) as shown in fig 1.

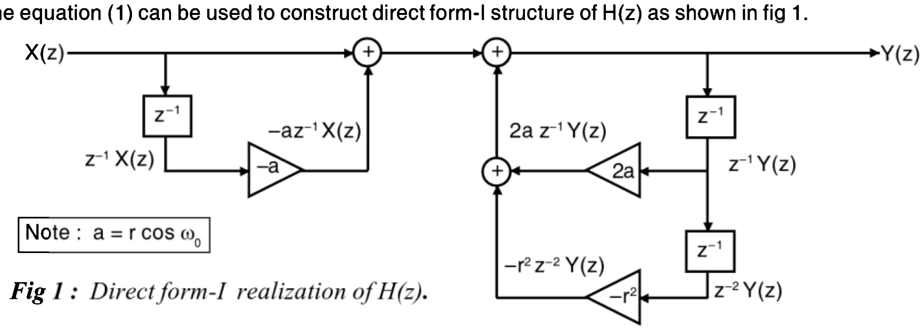


***Fig 1 :** Direct form-I realization of H(z).*

Consider the direct form-I structure as cascade of two systems $H_1(z)$ and $H_2(z)$ as shown in fig 2.

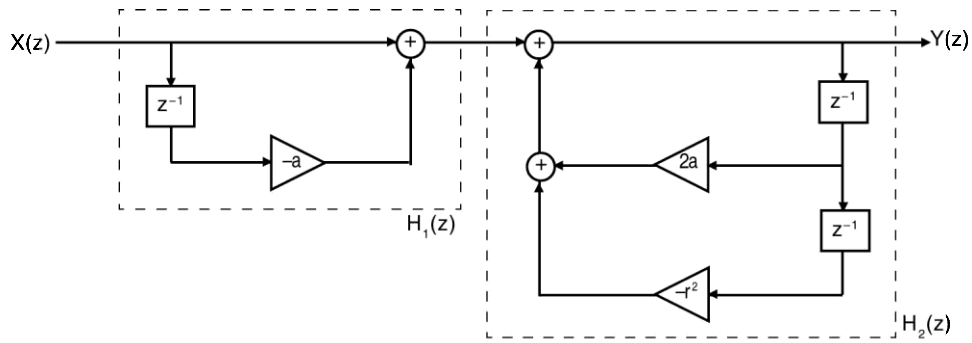


***Fig 2 :** Direct form-I structure as cascade of two systems.*

In an LT1 system, by linearity property, the order of cascading can be changed. Hence the systems $H_1(z)$ and $H_2(z)$ are interchanged and the fig 2 is redrawn as shown in fig 3.

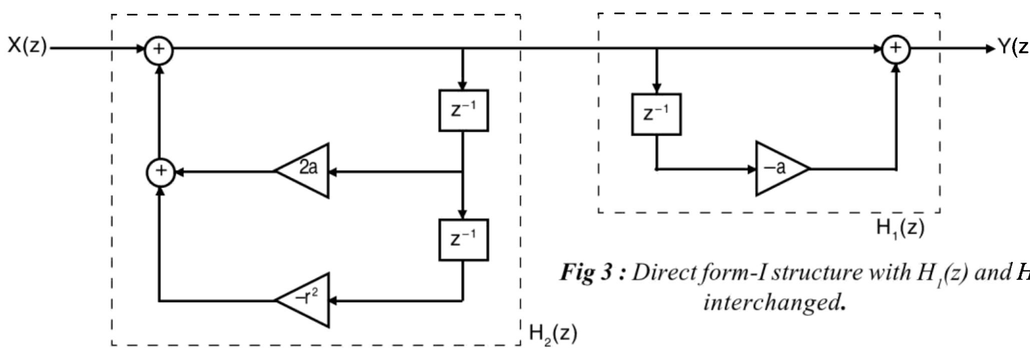


***Fig 3 :** Direct form-I structure with $H_1(z)$ and H_2 (z) interchanged.*

Since the input to delay elements in both the systems $H_1(z)$ and $H_2(z)$ are same, the outputs will also be same. Hence the delays can be combined and the resultant structure is direct form-II structure, which is shown in fig 4.

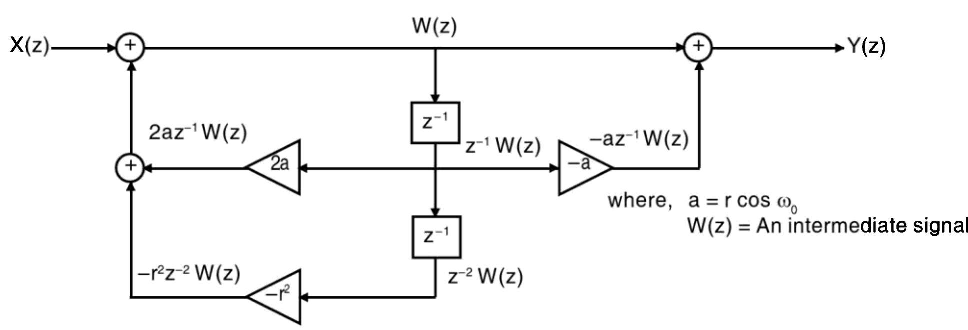


***Fig 4 :** Direct form-II structure of H(z).*

Example 10.5

Realize the given system in cascade and parallel forms.

$$H(z) = \frac{1 + \frac{1}{2}z^{-1}}{\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)}$$

Solution

Cascade Form

Let us realize the system as cascade of two second order systems.

$$H(z) = \frac{1 + \frac{1}{2}z^{-1}}{\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{1}{1 - z^{-1} + \frac{1}{4}z^{-2}} \times \frac{1 + \frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Let, $H(z) = H_1(z) \times H_2(z)$

where, $H_1(z) = \dfrac{1}{1 - z^{-1} + \frac{1}{4}z^{-2}}$; $H_2(z) = \dfrac{1 + \frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$

Let, $$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1 - z^{-1} + \frac{1}{4}z^{-2}} \qquad .....(1)$$

On cross multiplying equation (1) we get,

$$Y_1(z) - z^{-1}Y_1(z) + \frac{1}{4}z^{-2}Y_1(z) = X(z)$$

$$\therefore Y_1(z) = X(z) + z^{-1}Y_1(z) - \frac{1}{4}z^{-2}Y_1(z) \qquad .....(2)$$

The equation (2) can be realized in direct form-II structure as shown in fig 1.

Let, $$H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1 + \frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Let, $$\frac{Y(z)}{Y_1(z)} = \frac{W_2(z)}{Y_1(z)}\,\frac{Y(z)}{W_2(z)}$$

where, $$\frac{W_2(z)}{Y_1(z)} = \frac{1}{1 - z^{-1} + \frac{1}{2}z^{-2}} \qquad .....(3)$$

$$\frac{Y(z)}{W_2(z)} = 1 + \frac{1}{2}z^{-1} \qquad .....(4)$$

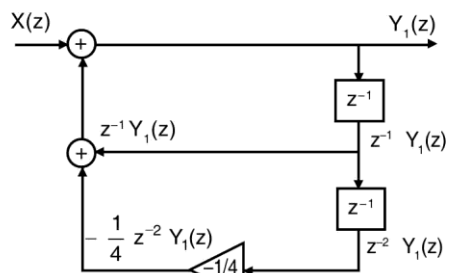


***Fig 1 :** Direct form-II structure of system $H_1(z)$.*

On cross multiplying equation (3) we get

$$W_2(z) - z^{-1}W_2(z) + \frac{1}{2}z^{-2}W_2(z) = Y_1(z)$$

$$\therefore\ W_2(z) = Y_1(z) + z^{-1}W_2(z) - \frac{1}{2}z^{-2}W_2(z) \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(z) = W_2(z) + \frac{1}{2}z^{-1}W_2(z) \qquad .....(6)$$

Using equations (5) and (6) the system $H_2(z)$ can be realized in direct form-II structure as shown in fig 2.

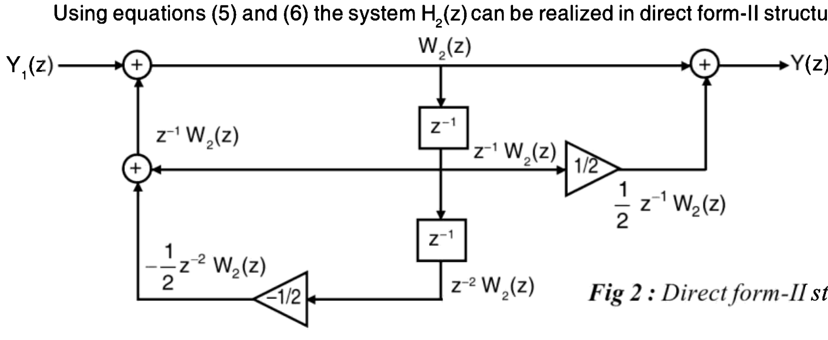


***Fig 2 :** Direct form-II structure of system $H_2(z)$.*

Cascade structure of H(z) is obtained by connecting structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

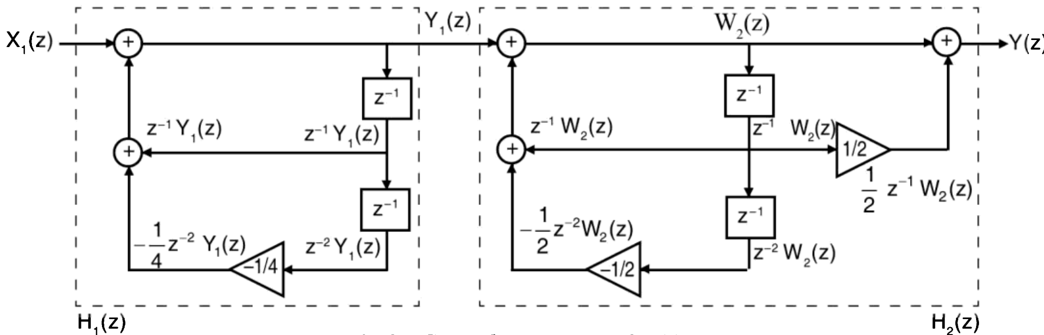


***Fig 3 :** Cascade structure of H(z).*

Parallel Realization

Given that, $H(z) = \dfrac{1+\frac{1}{2}z^{-1}}{\left(1-z^{-1}+\frac{1}{4}z^{-2}\right)\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)}$

By partial fraction expansion we can write,

$$H(z) = \frac{1+\frac{1}{2}z^{-1}}{\left(1-z^{-1}+\frac{1}{4}z^{-2}\right)\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)} = \frac{A+Bz^{-1}}{1-z^{-1}+\frac{1}{4}z^{-2}} + \frac{C+Dz^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}} \quad .....(7)$$

On cross multiplying equation (7) we get,

$$1+\frac{1}{2}z^{-1} = (A+Bz^{-1})\left(1-z^{-1}+\frac{1}{2}z^{-2}\right) + (C+Dz^{-1})\left(1-z^{-1}+\frac{1}{4}z^{-2}\right)$$

$$1+\frac{1}{2}z^{-1} = A-Az^{-1}+\frac{1}{2}Az^{-2}+Bz^{-1}-Bz^{-2}+\frac{1}{2}Bz^{-3}$$
$$+C-Cz^{-1}+\frac{1}{4}Cz^{-2}+Dz^{-1}-Dz^{-2}+\frac{1}{4}Dz^{-3}$$

$$1+\frac{1}{2}z^{-1} = (A+C)+(-A+B-C+D)z^{-1}+\left(\frac{1}{2}A-B+\frac{1}{4}C-D\right)z^{-2}$$
$$+\left(\frac{1}{2}B+\frac{1}{4}D\right)z^{-3} \quad .....(8)$$

On equating the constants in equation (8) we get,

$$A+C = 1 \quad \Rightarrow \quad C = 1-A$$

On equating the coefficients of z^{-3} in equation (8) we get,

$$\frac{1}{2}B+\frac{1}{4}D = 0 \qquad \therefore \ \frac{1}{4}D = -\frac{1}{2}B \qquad \Rightarrow \qquad D = -2B$$

On equating the coefficients of z^{-1} in equation (8) we get,

$$-A + B - C + D = \frac{1}{2}$$

On substituting C = 1 – A and D = –2B in the above equation we get,

$$-A + B - (1 - A) + (-2B) = \frac{1}{2} \quad \Rightarrow \quad -B = \frac{1}{2} + 1 \quad \Rightarrow \quad B = \frac{-3}{2}$$

$$\therefore \quad D = -2B = -2 \times \left(\frac{-3}{2}\right) = 3$$

On equating the coefficients of z^{-2} in equation (8) we get,

$$\frac{1}{2}A - B + \frac{1}{4}C - D = 0$$

On substituting $B = -3/2$, $C = 1 - A$, and $D = 3$ in the above equation we get,

$$\frac{1}{2}A - \left(-\frac{3}{2}\right) + \frac{1}{4}(1 - A) - 3 = 0 \quad \Rightarrow \quad \frac{1}{2}A - \frac{1}{4}A = -\frac{3}{2} - \frac{1}{4} + 3$$

$$\therefore \quad \frac{2A - A}{4} = \frac{-6 - 1 + 12}{4} \quad \Rightarrow \quad \frac{A}{4} = \frac{5}{4} \quad \Rightarrow \quad A = 5$$

$$\therefore C = 1 - A = 1 - 5 = -4$$

$$\therefore \; H(z) = \frac{A + Bz^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}} + \frac{C + Dz^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} = \frac{5 - \frac{3}{2}z^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}} + \frac{-4 + 3z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

$$\text{Let, } H(z) = \frac{5 - \frac{3}{2}z^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}} + \frac{-4 + 3z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} = H_1(z) + H_2(z)$$

$$\text{where, } H_1(z) = \frac{5 - \frac{3}{2}z^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}}$$

$$H_2(z) = \frac{-4 + 3z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)}; \quad H_1(z) = \frac{Y_1(z)}{X(z)}; \quad H_2(z) = \frac{Y_2(z)}{X(z)}$$

$$\therefore \; H(z) = H_1(z) + H_2(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{Y_1(z)}{X(z)} + \frac{Y_2(z)}{X(z)}$$

$$\therefore Y(z) = Y_1(z) + Y_2(z)$$

Realization of $H_1(z)$

$$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{5 - \frac{3}{2}z^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}}$$

Let, $\dfrac{Y_1(z)}{X(z)} = \dfrac{W_1(z)}{X(z)}\ \dfrac{Y_1(z)}{W_1(z)}$

where, $$\frac{W_1(z)}{X(z)} = \frac{1}{1 - z^{-1} + \frac{1}{4}z^{-2}} \qquad .....(9)$$

$$\frac{Y_1(z)}{W_1(z)} = 5 - \frac{3}{2}z^{-1} \qquad .....(10)$$

On cross multiplying equation (9) we get,

$$W_1(z) - z^{-1}W_1(z) + \frac{1}{4}z^{-2}W_1(z) = X(z)$$

$$\therefore W_1(z) = X(z) + z^{-1}W_1(z) - \frac{1}{4}z^{-2}W_1(z) \qquad .....(11)$$

On cross multiplying equation (10) we get,

$$Y_1(z) = 5W_1(z) - \frac{3}{2}z^{-1}W_1(z) \qquad .....(12)$$

The direct form-II structure of system $H_1(z)$ can be realized using equations (11) and (12) as shown in fig 4.

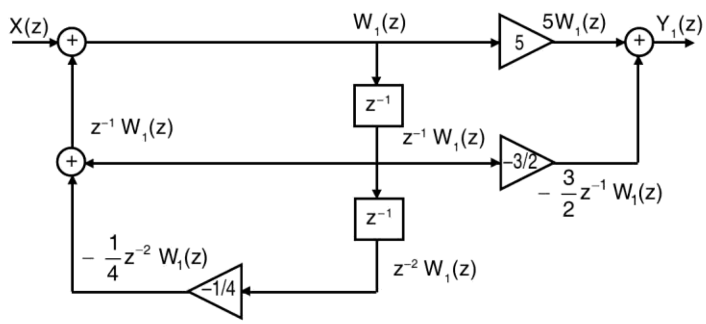


Fig 4 : Direct form-II structure of $H_1(z)$.

Realization of $H_2(z)$

$$H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{-4 + 3z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Let, $\dfrac{Y_2(z)}{X(z)} = \dfrac{W_2(z)}{X(z)}\ \dfrac{Y_2(z)}{W_2(z)}$

where, $$\frac{W_2(z)}{X(z)} = \frac{1}{1 - z^{-1} + \frac{1}{2}z^{-2}} \qquad .....(13)$$

$$\frac{Y_2(z)}{W_2(z)} = -4 + 3z^{-1} \qquad .....(14)$$

On cross multiplying the equation (13) we get,

$$W_2(z) - z^{-1}W_2(z) + \frac{1}{2}z^{-2}W_2(z) = X(z)$$

$$\therefore W_2(z) = X(z) + z^{-1}W_2(z) - \frac{1}{2}z^{-2}W_2(z) \qquad .....(15)$$

On cross multiplying equation (14) we get,

$$Y_2(z) = -4\,W_2(z) + 3z^{-1}W_2(z) \qquad .....\ (16)$$

The direct form-II structure of system $H_2(z)$ can be realized using equations (15) and (16) as shown in fig 5.

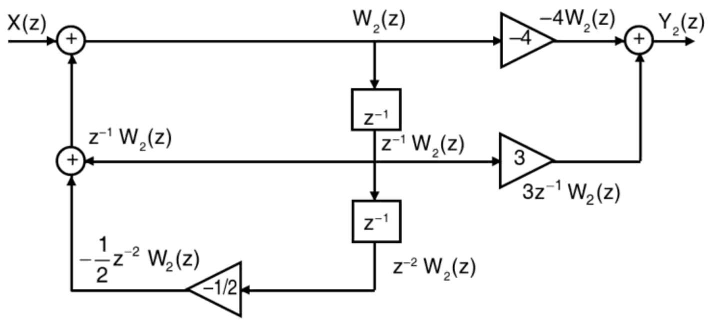


Fig 5 : Direct form-II structure of $H_2(z)$.

The parallel form structure of H(z) is obtained by connecting the direct form-II structure of $H_1(z)$ and $H_2(z)$ in parallel as shown in fig 6.

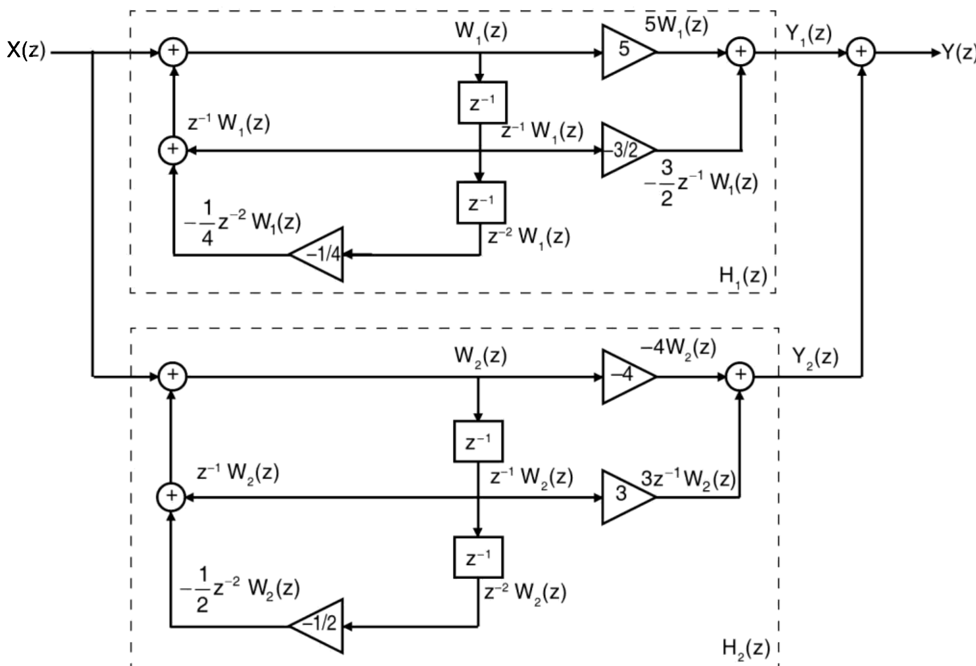


Fig 6 : Parallel form realization of system H(z).

Example 10.6

Obtain the cascade realization of the system, $H(z) = \dfrac{2 + z^{-1} + z^{-2}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}$

Solution

$$\text{Given that, } H(z) = \frac{2 + z^{-1} + z^{-2}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}$$

On examining the roots of numerator polynomial it is found that the roots are complex conjugate. Hence H(z) can be realized on cascade of one first order and one second order system.

$$\therefore\ H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \times \frac{2+z^{-1}+z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} = \frac{1}{1-\frac{1}{2}z^{-1}} \times \frac{2+z^{-1}+z^{-2}}{1+z^{-1}+\frac{1}{4}z^{-2}}$$

Let, $H(z) = H_1(z) \times H_2(z)$

$$\text{where, } H_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad \text{and} \quad H_2(z) = \frac{2+z^{-1}+z^{-2}}{1+z^{-1}+\frac{1}{4}z^{-2}}$$

$$\text{Let, } H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad .....(1)$$

On cross multiplying equation (1) we get,

$$Y_1(z) - \frac{1}{2}z^{-1}Y_1(z) = X(z)\ ; \quad \therefore Y_1(z) = X(z) + \frac{1}{2}z^{-1}Y_1(z) \qquad .....(2)$$

The direct form-II structure of $H_1(z)$ can be obtained from equation (2) as shown in fig 1.

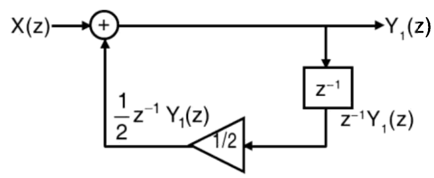


***Fig 1 :** Direct form-II structure of $H_1(z)$.*

$$\text{Let, } H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{2+z^{-1}+z^{-2}}{1+z^{-1}+\frac{1}{4}z^{-2}}$$

$$\text{Let, } \frac{Y(z)}{Y_1(z)} = \frac{W_2(z)}{Y_1(z)}\ \frac{Y(z)}{W_2(z)}$$

$$\text{where, } \frac{W_2(z)}{Y_1(z)} = \frac{1}{1+z^{-1}+\frac{1}{4}z^{-2}} \qquad .....(3)$$

$$\frac{Y(z)}{W_2(z)} = 2+z^{-1}+z^{-2} \qquad .....(4)$$

On cross multiplying equation (3) we get,

$$W_2(z) + z^{-1}W_2(z) + \frac{1}{4}z^{-2}W_2(z) = Y_1(z)$$

$$\therefore\ W_2(z) = Y_1(z) - z^{-1}W_2(z) - \frac{1}{4}z^{-2}W_2(z) \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(z) = 2W_2(z) + z^{-1}W_2(z) + z^{-2}W_2(z) \qquad ..... (6)$$

The direct form-II structure of $H_2(z)$ can be obtained using equations (5) and (6) as shown in fig 2.

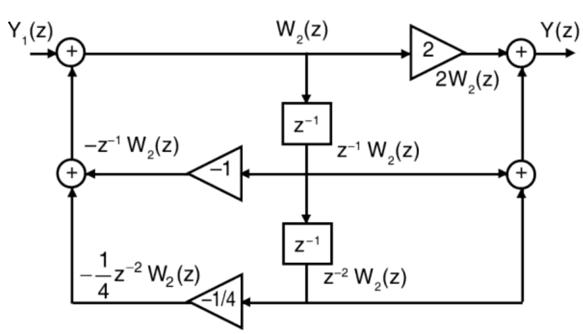


***Fig 2 :** Direct form-II structure of $H_2(z)$.*

The cascade realization of H(z) is obtained by connecting the direct form-II structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

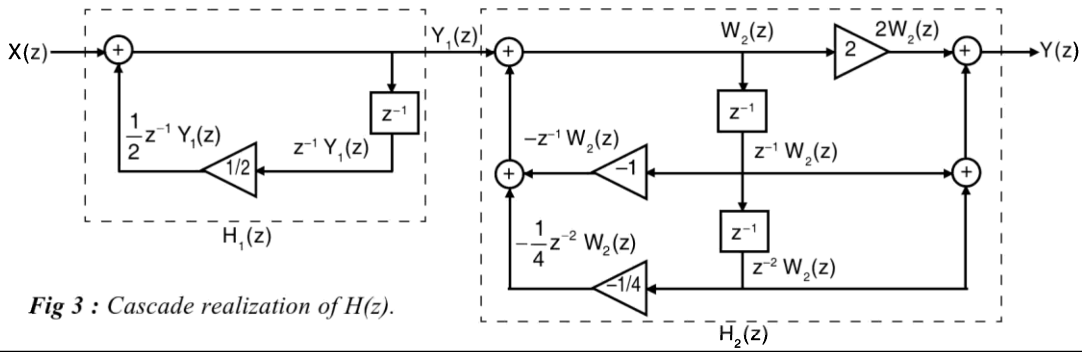


Fig 3 : *Cascade realization of H(z).*

Example 10.7

The transfer function of a system is given by, $H(z) = \dfrac{(1+z^{-1})^3}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)}$

Realize the system in cascade and parallel structures.

Solution

Cascade Realization

$$\text{Given that } H(z) = \frac{(1+z^{-1})^3}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)}$$

On examining the roots of the quadratic factor in the denominator it is observed that the roots are complex conjugate. Hence the system has to be realized as cascading of one first order section and one second order section.

$$\therefore H(z) = \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}} \times \frac{(1+z^{-1})^2}{1-z^{-1}+\frac{1}{2}z^{-2}} = \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}} \times \frac{1+2z^{-1}+z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

Let, $H(z) = H_1(z) \times H_2(z)$

$$\text{where, } H_1(z) = \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}} \quad \text{and} \quad H_2(z) = \frac{1+2z^{-1}+z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

$$\text{Let, } H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}} \qquad \text{..... (1)}$$

On cross multiplying equation (1) we get,

$$Y_1(z) - \frac{1}{4}z^{-1}Y_1(z) = X(z) + z^{-1}X(z)$$

$$\therefore Y_1(z) = X(z) + z^{-1}X(z) + \frac{1}{4}z^{-1}Y_1(z) \qquad \text{..... (2)}$$

The direct form-I structure of $H_1(z)$ can be drawn using equation (2) as shown in fig 1.

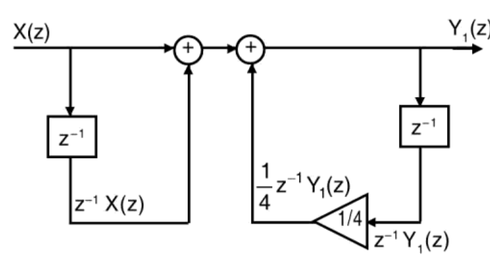


Fig 1 : *Direct form-I realization of $H_1(z)$.*

$$\text{Let, } H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}} \qquad .....(3)$$

On cross multiplying equation (3) we get

$$Y(z) - z^{-1}\,Y(z) + \frac{1}{2}z^{-2}\,Y(z) = Y_1(z) + 2z^{-1}\,Y_1(z) + z^{-2}\,Y_1(z)$$

$$\therefore\; Y(z) = Y_1(z) + 2z^{-1}\,Y_1(z) + z^{-2}\,Y_1(z) + z^{-1}\,Y(z) - \frac{1}{2}\,z^{-2}\,Y(z) \qquad .....(4)$$

The direct form-I structure of $H_2(z)$ can be drawn using equation (4) as shown in fig 2.

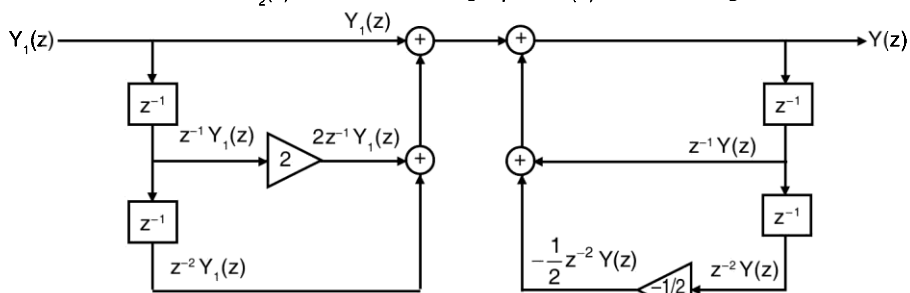


Fig 2 : The direct form-I structure of $H_2(z)$.

The cascade realization of H(z) is obtained by connecting the direct form - I structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

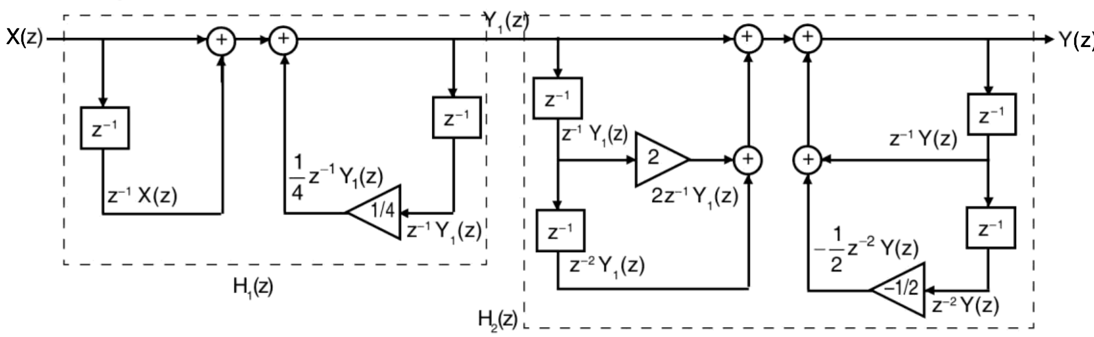


Fig 3 : Cascade realization of H(z).

Parallel Realization

$$\text{Given that, } H(z) = \frac{(1+z^{-1})^3}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{(1+z^{-1})(1+z^{-1})^2}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)}$$

$$= \frac{(1+z^{-1})(1 + 2z^{-1} + z^{-2})}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{1 + 2z^{-1} + z^{-2} + z^{-1} + 2z^{-2} + z^{-3}}{1 - z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{8}z^{-3}}$$

$$= \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}}$$

$$= 1 + \frac{\frac{17}{4}z^{-1} + \frac{9}{4}z^{-2} + \frac{9}{8}z^{-3}}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}}$$

$$\begin{array}{r|l} & \quad 1 \\ 1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3} & 1 + 3z^{-1} + 3z^{-2} + 2z^{-3} \\ & 1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3} \\ & (-) \quad (+) \quad\quad (-) \quad\quad (+) \\ \hline & \quad \frac{17}{4}z^{-1} + \frac{9}{4}z^{-2} + \frac{9}{8}z^{-3} \end{array}$$

$$\therefore\ H(z) = 1 + \frac{\frac{17}{4}z^{-1} + \frac{9}{4}z^{-2} + \frac{9}{8}z^{-3}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = 1 + z^{-1}\left[\frac{\frac{17}{4} + \frac{9}{4}z^{-1} + \frac{9}{8}z^{-2}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)}\right] \quad .....(5)$$

By partial fraction expansion we can write,

$$\frac{\frac{17}{4} + \frac{9}{4}z^{-1} + \frac{9}{8}z^{-2}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B + Cz^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} \quad .....(6)$$

On cross multiplying equation (6) we get,

$$\frac{17}{4} + \frac{9}{4}z^{-1} + \frac{9}{8}z^{-2} = A\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right) + (B + Cz^{-1})\left(1 - \frac{1}{4}z^{-1}\right) \quad .....(7)$$

$$\frac{17}{4} + \frac{9}{4}z^{-1} + \frac{9}{8}z^{-2} = A - Az^{-1} + \frac{1}{2}Az^{-2} + B - \frac{1}{4}Bz^{-1} + Cz^{-1} - \frac{1}{4}z^{-2} \quad .....(8)$$

The residue A can be solved by putting, $z^{-1} = 4$, in equation (7) as shown below.

$$\frac{17}{4} + \frac{9}{4}\times 4 + \frac{9}{8}\times 4^2 = A\left(1 - 4 + \frac{1}{2}\times 4^2\right) \Rightarrow \frac{17}{4} + 9 + 18 = A(1 - 4 + 8)$$

$$\therefore\ \frac{17 + 36 + 72}{4} = 5A \Rightarrow \frac{125}{4} = 5A$$

$$\therefore\ A = \frac{125}{4}\times\frac{1}{5} = \frac{25}{A}$$

On equating the constants in equation (8) we get,

$$A + B = \frac{17}{4} \Rightarrow B = \frac{17}{4} - A = \frac{17}{4} - \frac{25}{4} = -\frac{8}{4} = -2$$

On equating the coefficients of z^{-1} in equation (8) we get,

$$-A - \frac{1}{4}B + C = \frac{9}{4} \Rightarrow C = \frac{9}{4} + A + \frac{1}{4}B = \frac{9}{4} + \frac{25}{4} - \frac{2}{4} = \frac{32}{4} = 8$$

From equations (5) and (6) we can write,

$$H(z) = 1 + z^{-1}\left[\frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B + Cz^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}\right]$$

$$\therefore\ H(z) = 1 + \frac{\frac{25}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}} + \frac{-2z^{-1} + 8z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Let, $$H(z) = 1 + \frac{\frac{25}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}} + \frac{-2z^{-1} + 8z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}} = 1 + H_1(z) + H_2(z)$$

where, $$H_1(z) = \frac{\frac{25}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}}\ ; \qquad H_2(z) = \frac{-2z^{-1} + 8z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Let, $H(z) = \frac{Y(z)}{X(z)}$; $H_1(z) = \frac{Y_1(z)}{X(z)}$; $H_2(z) = \frac{Y_2(z)}{X(z)}$

$$\therefore H(z) = 1 + H_1(z) + H_2(z) \quad \Rightarrow \quad \frac{Y(z)}{X(z)} = 1 + \frac{Y_1(z)}{X(z)} + \frac{Y_2(z)}{X(z)}$$

$$\therefore Y(z) = X(z) + Y_1(z) + Y_2(z)$$

Realization of $H_1(z)$

$$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{\frac{25}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}}$$

On cross multiplying the above equation we get,

$$Y_1(z) - \frac{1}{4}z^{-1}\,Y_1(z) = \frac{25}{4}z^{-1}\,X(z)$$

$$\therefore \; Y_1(z) = \frac{25}{4}z^{-1}\,X(z) + \frac{1}{4}z^{-1}\,Y_1(z) \qquad \text{..... (9)}$$

Using equation (9) the direct form-I structure of $H_1(z)$ is drawn as shown in fig 4.

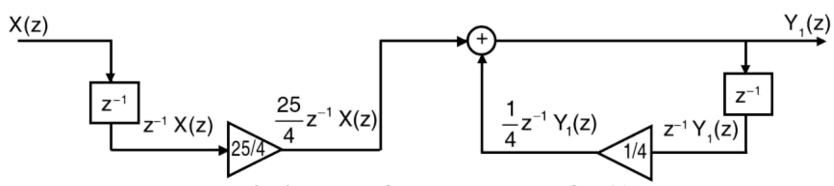


***Fig 4 :** Direct form-I structure of $H_1(z)$.*

Realization of $H_2(z)$

$$H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{-2z^{-1} + 8z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

On cross multiplying the above equation we get,

$$Y_2(z) - z^{-1}\,Y_2(z) + \frac{1}{2}z^{-2}\,Y_2(z) = -2z^{-1}\,X(z) + 8z^{-2}\,X(z)$$

$$\therefore \; Y_2(z) = -2z^{-1}\,X(z) + 8z^{-2}\,X(z) + z^{-1}\,Y_2(z) - \frac{1}{2}z^{-2}\,Y_2(z) \qquad \text{.....(10)}$$

Using equation (10) the direct form-I structure of $H_2(z)$ is drawn as shown in fig 5.

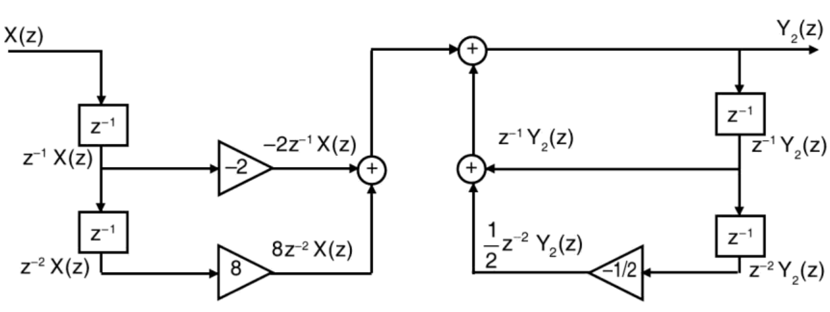


***Fig 5 :** Direct form-I structure of $H_2(z)$.*

Parallel Structure

The parallel structure of H(z) is obtained by connecting the direct form-I structure of $H_1(z)$ and $H_2(z)$ as shown in fig 6.

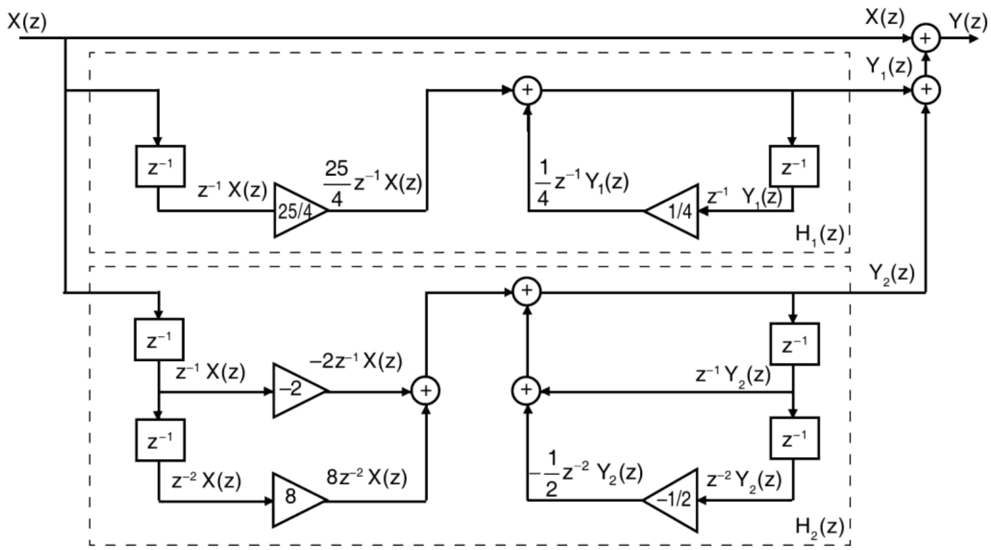


***Fig 6 :** The parallel structure of H(z).*

Example 10.8

An LTI System is described by the equation, $y(n) + y(n-1) - \frac{1}{4}y(n-2) = x(n)$.

Determine the cascade realization structure of the system.

Solution

Given that, $y(n) + y(n-1) - \frac{1}{4}y(n-2) = x(n)$

On taking $\mathcal{Z}$-transform we get,

$$Y(z) + z^{-1}Y(z) - \frac{1}{4}z^{-2}Y(z) = X(z)$$

$$\left(1 + z^{-1} - \frac{1}{4}z^{-2}\right)Y(z) = X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1}{1 + z^{-1} - \frac{1}{4}z^{-2}}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + z^{-1} - \frac{1}{4}z^{-2}} = \frac{1}{z^{-2}\left(z^2 + z - \frac{1}{4}\right)}$$

$$= \frac{1}{z^{-2}(z - 0.207)(z + 1.207)} = \frac{1}{(1 - 0.207z^{-1})(1 + 1.207z^{-1})}$$

$$z^2 + z - \frac{1}{4} = 0$$

$$z = \frac{-1 \pm \sqrt{1 + 4x\frac{1}{4}}}{2} = \frac{-1 \pm \sqrt{2}}{2} = +0.207 \text{ or } -1.207$$

Let, $H(z) = H_1(z)\,H_2(z)$

where, $H_1(z) = \frac{1}{1 - 0.207z^{-1}}$; $\qquad H_2(z) = \frac{1}{1 + 1.207z^{-1}}$

$$\text{Let, } H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1 - 0.207z^{-1}} \qquad .....(1)$$

On cross multiplying equation (1) we get,

$$Y_1(z) - 0.207z^{-1}\, Y_1(z) = X(z)$$

$$\therefore\ Y_1(z) = X(z) + 0.207z^{-1}\, Y_1(z) \qquad .....(2)$$

The direct form-I structure of $H_1(z)$ is obtained using equation (2) as shown in fig 1.

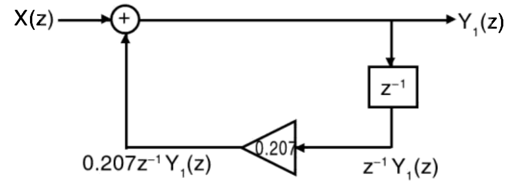


Fig 1 : Direct form-I structure of $H_1(z)$.

$$\text{Let, } H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1}{1 + 1.207z^{-1}} \qquad .....(3)$$

On cross multiplying equation (3) we get,

$$Y(z) + 1.207\, z^{-1}\, Y(z) = Y_1(z)$$

$$Y(z) = Y_1(z) - 1.207\, z^{-1}\, Y(z) \qquad .....(4)$$

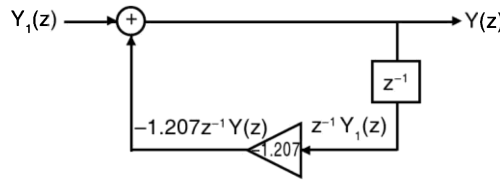


Fig 2 : Direct form-I structure of $H_2(z)$.

The direct form-I structure of $H_2(z)$ is obtained using equation (4) as shown in fig 2.The cascade structure is obtained by connecting the direct form structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

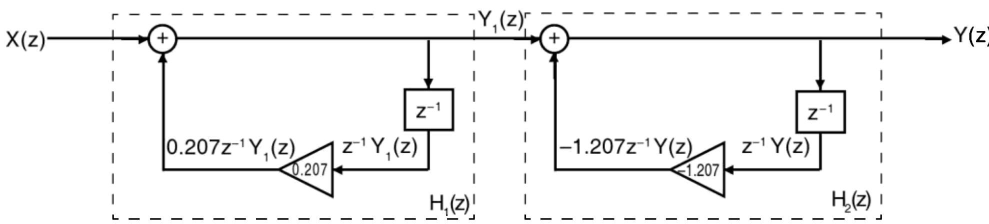


Fig 3 : Cascade structure.

10.4 Structures for Realization of FIR Systems

In general, the time domain representation of an N^{th} order FIR system is,

$$y(n) = \sum_{m=0}^{N-1} b_m\, x(n-m) = b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + ... + b_{N-1}\, x(n-(N-1))$$

and the z-domain representation of a FIR system is,

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1\, z^{-1} + b_2\, z^{-2} + ... + b_{N-1}\, z^{-(N-1)}$$

The above two representations of FIR system can be viewed as a computational procedure (or algorithm) to determine the output sequence y(n) from the input sequence x(n). These equations can be used to construct the block diagram of the system using delays, adders and multipliers. This block diagram is referred to as realization of the system or equivalently as a structure for realizing the system, (For block diagram representation of discrete system refer chapter - 6, section 6.6.2). Some of the block diagram representation of the system gives a direct relation between time domain equation and z-domain equation.

The different types of structures for realizing FIR systems are,

1. Direct form realization
2. Cascade realization
3. Linear phase realization

10.4.1 Direct Form Realization of FIR System

Consider the difference equation governing a FIR system,

$$y(n) = \sum_{m=0}^{N-1} b_m \, x(n-m)$$

$$= b_0 \, x(n) + b_1 \, x(n-1) + b_2 \, x(n-2) + \ldots\ldots + b_{N-1} \, x(n-(N-1))$$

If $\mathcal{Z}\{x(n)\} = X(z)$ then, $\mathcal{Z}\{x(n-k)\} = z^{-k} X(z)$

On taking $\mathcal{Z}$-transform of the above equation we get,.

$$\therefore \; Y(z) = b_0 \, X(z) + b_1 \, z^{-1} \, X(z) + b_2 \, z^{-2} \, X(z) + b_3 \, z^{-3} \, X(z) + \ldots\ldots + b_{N-2} \, z^{-(N-2)} \, X(z) + b_{N-1} \, z^{-(N-1)} \, X(z) \quad \ldots\ldots (10.12)$$

The equation of Y(z) [equation (10.12)] can be directly represented by a block diagram as shown in fig 10.8 and this structure is called direct form structure. The direct form structure provides a direct relation between time domain and z-domain equations.

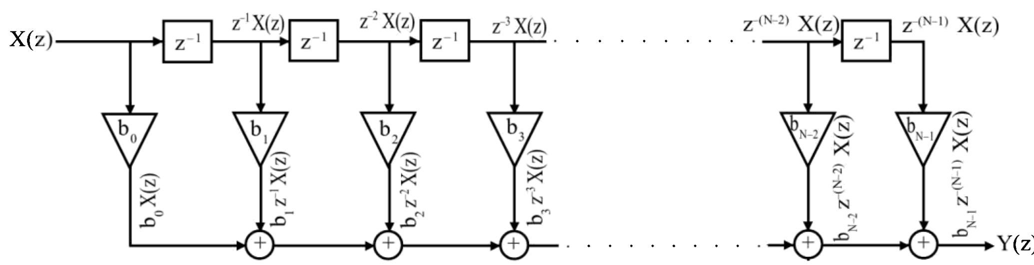


***Fig 10.8 :** Direct form structure of FIR system.*

From the direct form structure it is observed that the realization of an N^{th} order FIR discrete time system involves N number of multiplications and N–1 number of additions. Also the structure involves N–1 delays and so N–1 memory locations are required to store the delayed signals.

10.4.2 Cascade Form Realization of FIR System

Consider the transfer function of a FIR system,

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 \, z^{-1} + b_2 \, z^{-2} + \ldots\ldots + b_{N-1} \, z^{-(N-1)}$$

The transfer function of FIR system is $(N-1)^{th}$ order polynomial in z. This polynomial can be factorized into first and second order factors and the transfer function can be expressed as a product of first and second order factors or sections as shown in equation (10.13).

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) \times H_2(z) \times H_3(z) \ldots\ldots H_m(z) = \prod_{i=1}^{m} H_i(z) \quad \ldots\ldots(10.13)$$

where, $H_i(z) = c_{0i} + c_{1i} \, z^{-1} + c_{2i} \, z^{-2}$ — Second order section

or, $H_i(z) = c_{0i} + c_{1i} \, z^{-1}$ — First order section

The individual second order or first order sections can be realized either in direct form structure or linear phase structure. The overall system is obtained by cascading the individual sections as shown in fig 10.9. The number of calculations and the memory requirement depends on the realization of individual sections.

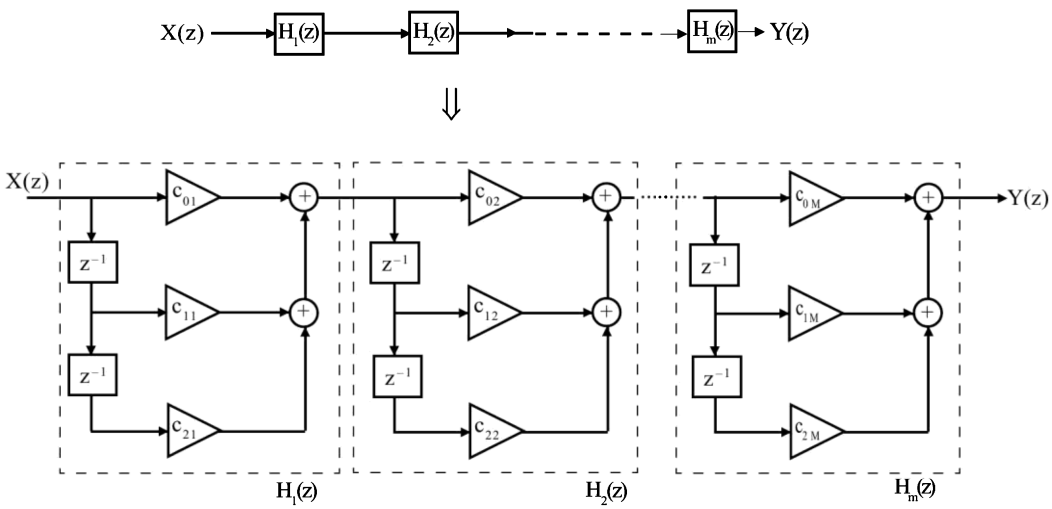


***Fig 10.9 :** Cascade structure of FIR system.*

10.4.3 Linear Phase Realization of FIR System

Consider the impulse response, h(n) of FIR system,

$$h(n) = \{\underset{\uparrow}{b_0},\ b_1,\ b_2,\ \dots\dots\dots\dots\ b_{N-1}\}$$

In FIR system, for linear phase response the impulse response should be symmetrical.

The condition for symmetry is,

$$h(n) = h(N-1-n)$$

Proof :

Let, N = 7, $\therefore$ h(n) = h(6–n)	Let, N = 8, $\therefore$ h(n) = h(7–n)
n = 0, 1, 2, 3, 4, 5, 6	n = 0, 1, 2, 3, 4, 5, 6, 7
When n = 0; h(0) = h(6)	When n = 0; h(0) = h(7)
When n = 1; h(1) = h(5)	When n = 1; h(1) = h(6)
When n = 2; h(2) = h(4)	When n = 2; h(2) = h(5)
When n = 3; h(3) = h(3)	When n = 3; h(3) = h(4)

When the impulse response is symmetric, the samples of impulse response will satisfy the condition,

$$b_n = b_{N-1-n}$$

By using the above symmetry condition it is possible to reduce the number of multipliers required for the realization of FIR system. Hence, the linear phase realization is also called ***realization with minimum number of multipliers.***

Consider the transfer function of a FIR system,

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots.. + b_{N-1} z^{-(N-1)}$$

The linear phase realization of the FIR system using the above equation for even and odd values of N are discussed below.

Case i : When N is even

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots.. + b_{N-1} z^{-(N-1)}$$

$$= \sum_{m=0}^{N-1} b_m z^{-m} = \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{m=\frac{N}{2}}^{N-1} b_m z^{-m}$$

Let, $p = N-1-m$, $\quad \therefore m = N-1-p$

When $m = \frac{N}{2}$; $p = N-1-\frac{N}{2} = \frac{N}{2}-1$

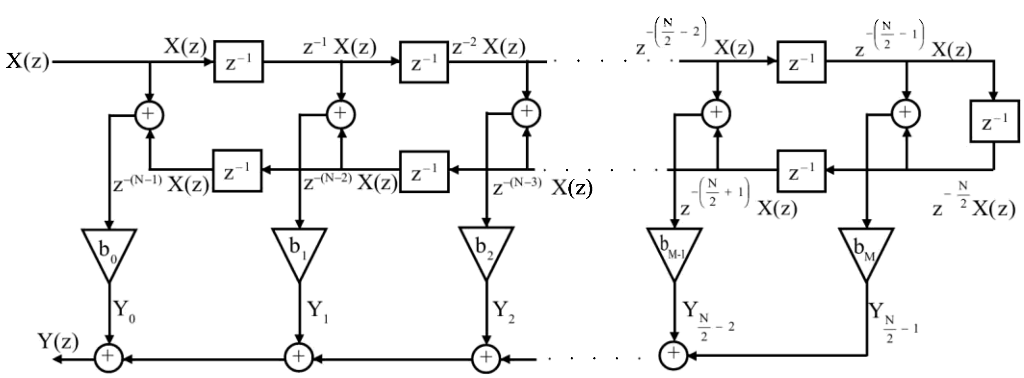
When $m = N-1$; $p = N-1-(N-1) = 0$

$$= \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{p=0}^{\frac{N}{2}-1} b_{N-1-p}\, z^{-(N-1-p)}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{m=0}^{\frac{N}{2}-1} b_{N-1-m}\, z^{-(N-1-m)} \qquad \boxed{\text{Let } p = m}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-(N-1-m)} \qquad \boxed{\because b_m = b_{N-1-m}}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} b_m \left[z^{-m} + z^{-(N-1-m)}\right]$$



where $M = \frac{N}{2}-1$; $\quad Y_0 = b_0\left[X(z) + z^{-(N-1)} X(z)\right]$; $\quad Y_{\frac{N}{2}-2} = b_{\frac{N}{2}-2}\left[z^{-\left(\frac{N}{2}-2\right)} X(z) + z^{-\left(\frac{N}{2}+1\right)} X(z)\right]$

$Y_1 = b_1\left[z^{-1} X(z) + z^{-(N-2)} X(z)\right]$; $\quad Y_{\frac{N}{2}-1} = b_{\frac{N}{2}-1}\left[z^{-\left(\frac{N}{2}-1\right)} X(z) + z^{-\frac{N}{2}} X(z)\right]$

Fig 10.10 : *Direct form realization of a linear phase FIR system when N is even.*

$$\therefore\ Y(z) = b_0\left[X(z) + z^{-(N-1)}X(z)\right] + b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right] + ...$$

$$+ b_{\frac{N}{2}-2}\left[z^{-\left(\frac{N}{2}-2\right)}X(z) + z^{-\left(\frac{N}{2}+1\right)}X(z)\right] + b_{\frac{N}{2}-1}\left[z^{-\left(\frac{N}{2}-1\right)}X(z) + z^{-\frac{N}{2}}X(z)\right]$$

When N is even, the above equation can be used to construct the direct form structure of linear phase FIR system with minimum number of multipliers, as shown in fig 10.10. From the direct form linear phase structure it is observed that the realization of an N^{th} order FIR discrete time system for even values of N involves N/2 number of multiplications and N–1 number of additions. Also the structure involves N–1 delays and so N–1 memory locations are required to store the delayed signals.

Case ii : When N is odd

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)} = \sum_{m=0}^{N-1} b_m z^{-m}$$

$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=\frac{N+1}{2}}^{N-1} b_m z^{-m}$$

Let, p = N–1–m, $\quad\therefore$ m = N–1–p

When $m = \dfrac{N+1}{2}$; $\quad p = N-1-\dfrac{N+1}{2} = \dfrac{N-3}{2}$

When m = N–1 ; p = N–1 – (N–1) = 0

$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{p=0}^{\frac{N-3}{2}} b_{N-1-p} z^{-(N-1-p)}$$

$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} b_{N-1-m} z^{-(N-1-m)} \qquad \boxed{\text{Let } p = m}$$

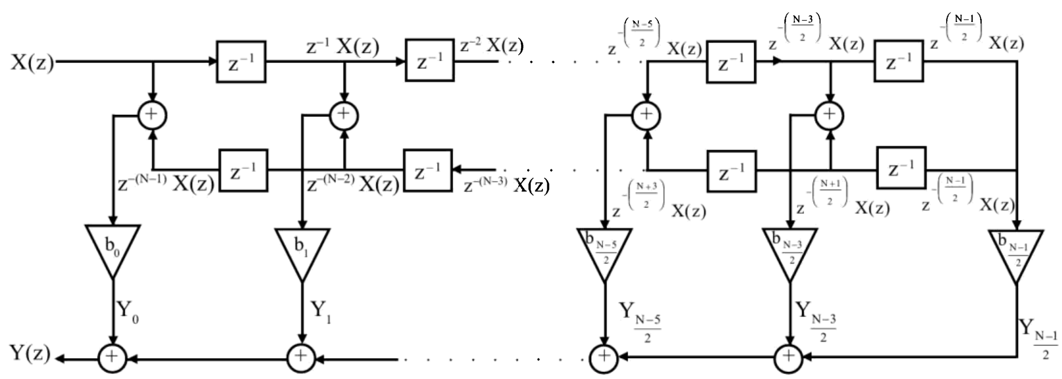
$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-(N-1-m)} \qquad \boxed{\because b_m = b_{N-1-m}}$$

$$= b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} b_m \left[z^{-m} + z^{-(N-1-m)}\right]$$

$$\therefore\ Y(z) = b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} X(z) + b_0\left[X(z) + z^{-(N-1))}X(z)\right] + b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right] +$$

$$..... + b_{\frac{N-5}{2}}\left[z^{-\left(\frac{N-5}{2}\right)}X(z) + z^{-\left(\frac{N+3}{2}\right)}X(z)\right] + b_{\frac{N-3}{2}}\left[z^{-\left(\frac{N-3}{2}\right)}X(z) + z^{-\left(\frac{N+1}{2}\right)}X(z)\right]$$

When N is odd, the above equation can be used to construct the direct form structure of linear phase FIR system with minimum number of multipliers, as shown in fig 10.11.



$$Y_0 = b_0\left[X(z) + z^{-(N-1)}X(z)\right]; \quad Y_{\frac{N-5}{2}} = b_{\frac{N-5}{2}}\left[z^{-\left(\frac{N-5}{2}\right)}X(z) + z^{-\left(\frac{N+3}{2}\right)}X(z)\right]; \quad Y_{\frac{N-1}{2}} = b_{\frac{N-1}{2}}\, z^{-\left(\frac{N-1}{2}\right)}X(z)$$

$$Y_1 = b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right]; \quad Y_{\frac{N-3}{2}} = b_{\frac{N-3}{2}}\left[z^{-\left(\frac{N-3}{2}\right)}X(z) + z^{-\left(\frac{N+1}{2}\right)}X(z)\right]$$

***Fig 10.11 :** Direct form realization of a linear phase FIR system when N is odd.*

From the direct form linear phase structure it is observed that the realization of an N^{th} order FIR discrete time system for odd values of N involves (N+1)/2 number of multiplications and N−1 number of additions. Also the structure involves N−1 delays and so N−1 memory locations are required to store the delayed signals.

Example 10.9

Draw the direct form structure of the FIR system described by the transfer function

$$H(z) = 1 + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{2}z^{-4} + \frac{1}{8}z^{-5}$$

Solution

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{2}z^{-4} + \frac{1}{8}z^{-5}$$

$$\therefore\ Y(z) = X(z) + \frac{1}{2}z^{-1}X(z) + \frac{3}{4}z^{-2}X(z) + \frac{1}{4}z^{-3}X(z) + \frac{1}{2}z^{-4}X(z) + \frac{1}{8}z^{-5}X(z) \quad .....(1)$$

The direct form structure of FIR system can be obtained directly from equation (1).

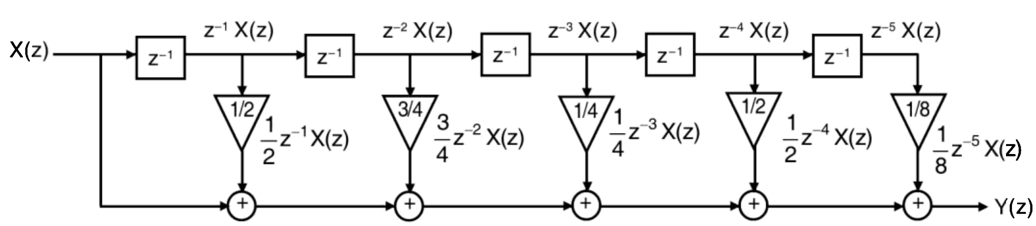


***Fig 1 :** Direct form structure of H(z).*

Example 10.10

Realize the following system with minimum number of multipliers.

a) $H(z) = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$

b) $H(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + z^{-3}$

c) $H(z) = \left(1 + \frac{1}{2}z^{-1} + z^{-2}\right)\left(1 + \frac{1}{4}z^{-1} + z^{-2}\right)$

Solution

a) Given that, $H(z) = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$ ..... (1)

By the definition of $\mathcal{Z}$-transform we get,

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n} = h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + \ldots.. \qquad \ldots..(2)$$

On comparing equations (1) and (2) we get,

Impulse response, $h(n) = \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right\}$

Here h(n) satisfies the condition $h(n) = h(N-1-n)$ and so impulse response is symmetrical. Hence the system has linear phase and can be realized with minimum number of multipliers.

Let, $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$

$$\therefore\ Y(z) = \frac{1}{4}X(z) + \frac{1}{2}z^{-1}X(z) + \frac{3}{4}z^{-2}X(z) + \frac{1}{2}z^{-3}X(z) + \frac{1}{4}z^{-4}X(z)$$

$$= \frac{1}{4}\left[X(z) + z^{-4}X(z)\right] + \frac{1}{2}\left[z^{-1}X(z) + z^{-3}X(z)\right] + \frac{3}{4}z^{-2}X(z) \qquad \ldots..(3)$$

The direct form structure of linear phase FIR system is constructed using equation (3) as shown in fig 1.

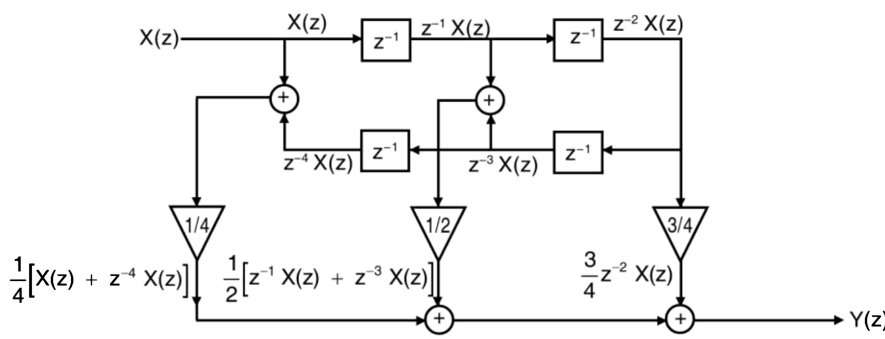


Fig 1 : Linear phase realization of H(z).

b) Given that, $H(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + z^{-3}$

Let, $H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + z^{-3}$

$$\therefore Y(z) = X(z) + \frac{1}{2}z^{-1}X(z) + \frac{1}{2}z^{-2}X(z) + z^{-3}X(z)$$

$$= \left[X(z) + z^{-3}X(z)\right] + \frac{1}{2}\left[z^{-1}X(z) + z^{-2}X(z)\right] \qquad \ldots..(4)$$

The direct form realization of H(z) with minimum number of multipliers (i.e., linear phase realization) is obtained using equation (4) as shown in fig 2.

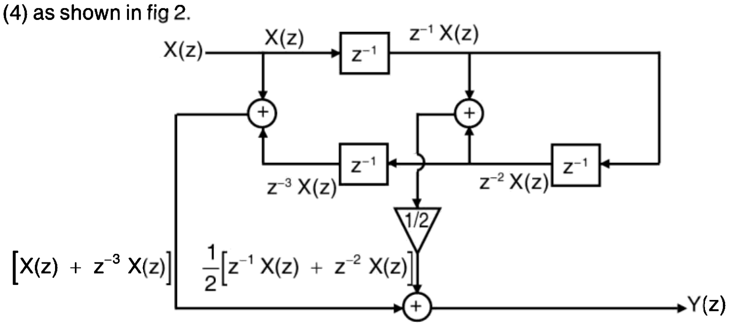


Fig 2 : Linear phase realization of H(z).

c) Given that, $H(z) = \left(1 + \frac{1}{2}z^{-1} + z^{-2}\right)\left(1 + \frac{1}{4}z^{-1} + z^{-2}\right)$

The given system can be realized as cascade of two second order systems. Each system can be realized with minimum number of multipliers.

Let, $H(z) = H_1(z)\, H_2(z)$

where, $H_1(z) = 1 + \frac{1}{2}z^{-1} + z^{-2}$; $H_2(z) = 1 + \frac{1}{4}z^{-1} + z^{-2}$

Let, $H_1(z) = \frac{Y_1(z)}{X(z)} = 1 + \frac{1}{2}z^{-1} + z^{-2}$

$$\therefore\ Y_1(z) = X(z) + \frac{1}{2}z^{-1} X(z) + z^{-2} X(z)$$

$$= \left[X(z) + z^{-2} X(z)\right] + \frac{1}{2}z^{-1} X(z) \qquad .....(5)$$

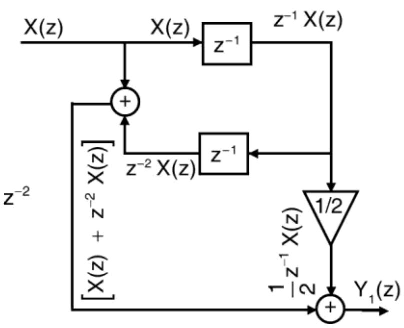


Fig 3 : Linear phase realization of $H_1(z)$.

The linear phase realization structure of $H_1(z)$ is obtained using equation (5) as shown in fig 3.

Let, $H_2(z) = \frac{Y(z)}{Y_1(z)} = 1 + \frac{1}{4}z^{-1} + z^{-2}$

$$\therefore\ Y(z) = Y_1(z) + \frac{1}{4}z^{-1} Y_1(z) + z^{-2} Y_1(z)$$

$$= \left[Y_1(z) + z^{-2} Y_1(z)\right] + \frac{1}{4}z^{-1} Y_1(z) \qquad .....(6)$$

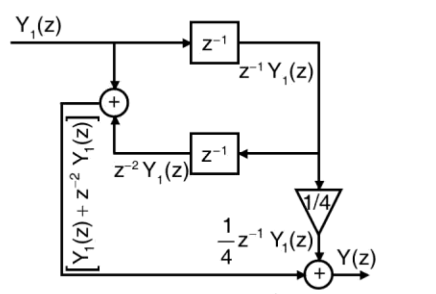


Fig 4 : Linear phase realization of $H_2(z)$.

The linear phase realization structure of $H_2(z)$ is obtained using equation (6) as shown in fig 4. The linear phase structure of H(z) is obtained by connecting the linear phase realization structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 5.

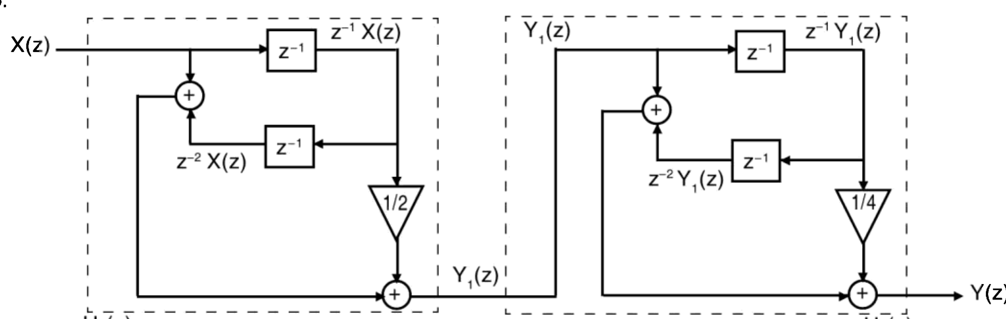


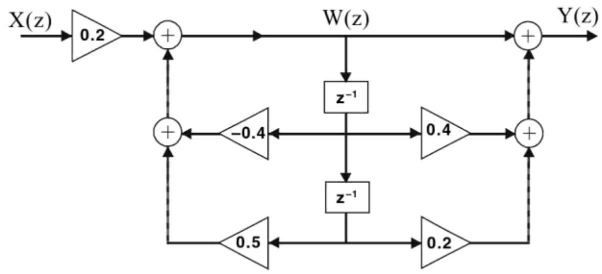
Fig 5 : Cascade realization of H(z).

10.5 Summary of Important Concepts

1. Mathematically, a discrete time system is represented by a difference equation.
2. Physically, a discrete time system is realized or implemented either as a digital hardware or as a software running on a digital hardware.
3. The processing of the discrete time signal by the digital hardware involves mathematical operations like addition, multiplication, and delay.
4. The time taken to process the discrete time signal and the computational complexity, depends on number of calculations involved and the type of arithmetic used for computation.
5. The various structures proposed for IIR and FIR systems, attempt to reduce the computational complexity, errors in computation and the memory requirement of the system.
6. When a discrete time system is designed by considering all the infinite samples of the impulse response, then the system is called IIR (Infinite Impulse Response) system.
7. When a discrete time system is designed by choosing only finite samples (usually N-samples) of the impulse response, then the system is called FIR (Finite Impulse Response) system.
8. The IIR systems are recursive systems, whereas the FIR systems are nonrecursive systems.
9. The direct form-I structure of IIR system offers a direct relation between time domain and z-domain equations.
10. Since separate delays are employed for input and output samples, realizing IIR system using direct form-I structure require more memory.
11. The direct form-I and II structure realization of an N^{th} order IIR discrete time system involves M+N+1 number of multiplications and M+N number of additions.
12. The direct form-I structure realization of an N^{th} order IIR discrete time system involves M+N delays and so M+N memory locations are required to store the delayed signals.
13. In a realizable N^{th} order IIR discrete time system, the direct form-II structure realization involves N delays and so N memory locations are required to store the delayed signals.
14. In canonic structure, the number of delays will be equal to the order of the system.
15. The direct form-II structure of IIR system is canonic whereas the direct form-I structure is noncanonic.
16. In cascade realization of IIR system, the N^{th} order transfer function is divided into first and second order sections and they are realized in direct form-I or II structure and then connected in cascade.
17. In parallel realization of IIR system, the N^{th} order transfer function is divided into first and second order sections and they are realized in direct form-I or II structure and then connected in parallel.
18. In cascade and parallel realization of IIR systems, the number of calculations and the memory requirement depends on the realization of individual sections.
19. Direct form structure of FIR system provides a direct relation between time domain and z-domain equations.
20. The realization of an N^{th} order FIR discrete time system using direct form structure and linear phase structure involves N number of multiplications and N–1 number of additions.
21. The realization of an N^{th} order FIR discrete time system using direct form structure involves N–1 delays and so N–1 memory locations are required to store the delayed signals.
22. The condition for symmetry of impulse response of FIR system is, $h(n) = h(N-1-n)$.
23. The linear phase realization is also called realization with minimum number of multipliers.
24. In cascade realization of FIR system, the N^{th} order transfer function is divided into first and second order sections and they are realized in direct form or linear phase structure and then connected in cascade.
25. The direct form linear phase realization structure of an N^{th} order FIR discrete time system for even values of N involves N/2 number of multiplications, and N–1 number of additions.
26. The direct form linear phase realization structure of an N^{th} order FIR discrete time system for odd values of N involves (N+1)/2 number of multiplications, and N–1 number of additions.

10. 6 Short Questions and Answers

Q10.1 ***Obtain the transfer function for the following structure.***



Solution

The following z-domain equations can be obtained from the given direct form-II structure.

$$W(z) = -0.4z^{-1}W(z) + 0.5z^{-2}W(z) + 0.2\,X(z)$$

$$\therefore W(z) + 0.4z^{-1}W(z) - 0.5z^{-2}W(z) = 0.2\,X(z) \quad \Rightarrow \quad \frac{W(z)}{X(z)} = \frac{0.2}{1+0.4\,z^{-1} - 0.5\,z^{-2}}$$

$$Y(z) = W(z) + 0.4\,z^{-1}W(z) + 0.2\,z^{-2}W(z) \quad \Rightarrow \quad \frac{Y(z)}{W(z)} = 1 + 0.4z^{-1} + 0.2z^{-2}$$

The given direct form-II digital network can be realized by the transfer function,

$$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.2\,(1 + 0.4z^{-1} + 0.2z^{-2})}{1 + 0.4z^{-1} - 0.5z^{-2}}$$

Q10.2 ***Realize the following FIR system with minimum number of multipliers.***
h(n) = { –0.5, 0.8, –0.5 }

Solution

Given that, h(n) = {–0.5, 0.8, –0.5 }

On taking Z- transform,

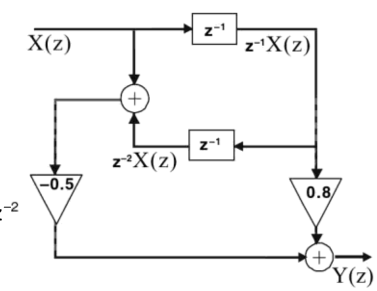
$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-1} = \sum_{n=0}^{2} h(n)z^{-1}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} = -0.5 + 0.8z^{-1} - 0.5z^{-2}$$

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = -0.5 + 0.8z^{-1} - 0.5z^{-2}$$

$$\therefore Y(z) = -0.5\,X(z) + 0.8z^{-1}X(z) - 0.5z^{-2}X(z)$$

$$= -0.5\,[\,X(z) + z^{-2}X(z)\,] + 0.8z^{-1}X(z)$$



FigQ10.2: *Linear phase realization.*

The linear phase structure is drawn using the above equation as shown in fig Q10.2.

Q10.3 ***The transfer function of an IIR system has 'Z' number of zeros and 'P' number of poles. How many number of additions, multiplications and memory locations are required to realize the system in direct form-I and direct form-II.***

The realization of IIR system with Z zeros and P poles in direct form-I and II structure, involves Z+P number of additions and Z+P+1 number of multiplications. The direct form-I structure requires Z+P memory locations whereas the direct form-II structure requires only P number of memory locations.

Q10.4 ***What are the factors that influence the choice of structure for realization of an LTI system?***

The factors that influence the choice of realization structure are computational complexity, memory requirements, finite word length effects, parallel processing and pipelining of computations.

Q10.5 ***Draw the direct form-I structure of second order IIR system with equal number of poles and zeros.***

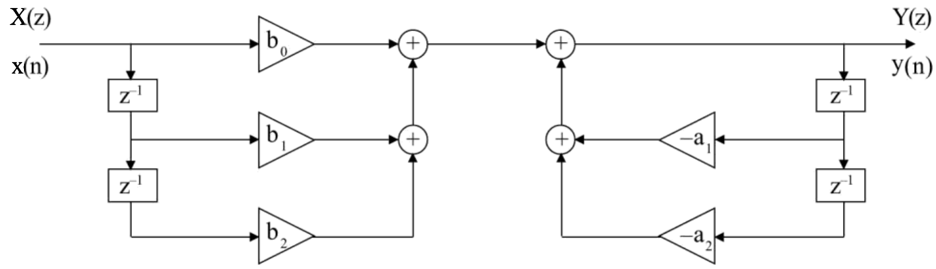


Fig Q10.5 : *Direct form-I structure of second order IIR system.*

Q10.6 ***An LTI system is described by the difference equation, $y(n) = a_1y(n-1) + x(n) + b_1x(n-1)$. Realize it in direct form-I structure and convert to direct form-II structure.***

Solution

Given that, $y(n) = a_1y(n-1) + x(n) + b_1x(n-1)$. Using the given equation the direct form-I structure is drawn as shown in fig Q10.6a.

Direct form-I structure can be considered as cascade of two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ as shown in fig Q10.6b. By linearity property, order of cascading can be changed as shown in fig Q10.6c.

In fig Q10.6c, we can observe that the input to the delay in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same and so the output of delays will be same. Hence the delays can be combined to get direct form-II structure as shown in fig Q10.6d.

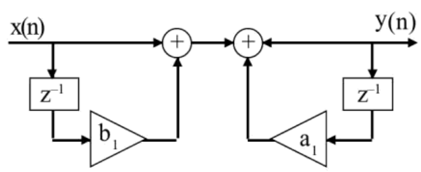


***Fig Q10.6a**: Direct form-I structure.*

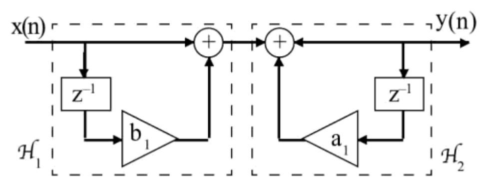


***Fig Q10.6b**: Direct form-I structure as cascade of two systems.*

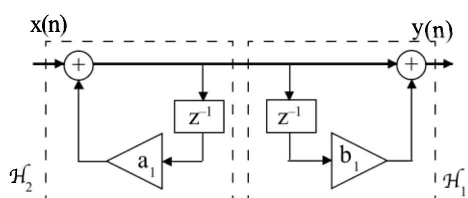


***Fig Q10.6c**: Direct form-I structure after interchanging the order of cascading.*

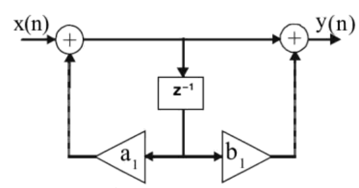


***Fig Q10.6d**: Direct form-II structure.*

Q10.7 ***What is the advantage in cascade and parallel realization of IIR systems ?***

In digital implementation of LTI system the coefficients of the difference equation governing the system are quantized. While quantizing the coefficients the value of poles may change. This will end up in a frequency response different to that of desired frequency response.

These effects can be avoided or minimized, if the LTI system is realized in cascade or parallel structure. [i.e, The sensitivity of frequency response characteristics to quantization of the coefficients is minimized]

Q10.8 Compare the direct form-I and II structures of an IIR systems, with M zeros and N poles.

Direct form-I	*Direct form-II*
1. Separate delay for input and output.	1. Same delay for input and output.
2. M + N + 1 multiplications are involved.	2. M + N + 1 multiplications are involved.
3. M + N additions are involved.	3. M + N additions are involved.
4. M + N delays are involved.	4. N delays are involved.
5. M + N memory locations are required.	5. N memory locations are required.
6. Noncanonical structure.	6. Canonical structure.

Q10.9 Compare the direct form and linear phase structures of an N^{th} order FIR system.

Direct form	*Linear phase*
1. Impulse response need not be symmetric.	1. Impulse response should be symmetric.
2. N multiplications are involved.	2. N/2 or (N+1)/2 multiplications are involved.
3. N−1 additions and delays are involved.	3. N−1 additions and delays are involved.
4. N−1 memory locations are required.	4. N−1 memory locations are required.

Q10.10 What is the advantage in linear phase realization of FIR systems ?

The advantage in the linear phase realization structure is that it involves minimum number of multiplications. In linear phase realization of N^{th} order FIR system, the number of multiplications for even values of N will be N/2 and for odd values of N will be (N+1)/2, whereas the direct form realization involves N multiplications.

10. 7 Exercises

I. Fill in the blanks with appropriate words.

1. In IIR systems, the ___________ structure will give direct relation between time domain and z-domain.
2. When number of delays is equal to order of the system, the structure is called ___________.
3. The direct form realization of IIR system with M zeros and N poles involves ___________ multiplications.
4. The direct form-II realization of N^{th} order IIR system requires ___________ delays and memory locations.
5. The direct form realization of N^{th} order FIR system involves ___________ additions.
6. ___________ realization is called realization with minimum number of multipliers.

Answers		
1. direct form-I	3. M+N+1	5. N−1
2. canonic structure	4. N	6. Linear phase

II. State whether the following statements are True/False.

1. The direct form-I structure of IIR system employs same delay for input and output samples.
2. In direct form-II realization of IIR system, N memory locations are required to store delayed signals.
3. In parallel or cascade realization, the memory requirement depends on realization of individual sections.
4. Scaling multipliers has to be provided between individual sections of cascade structure.
5. The linear phase realization of N^{th} order FIR system for odd values of N involves N/2 multiplications.
6. For linear phase realization of FIR system, the impulse response should be symmetric.

Answers					
1. False	2. True	3. True	4. True	5. False	6. True

III. Choose the right answer for the following questions

1. The factor that influence the choice of realization of strurcture is,

a) memory requirements b) computational complexity
c) parallel processing and pipelining d) all the above.

2. The structure that uses separate delays for input and output samples is,

a) direct form-II b) direct form-I c) cascade form d) parallel form

3. The linear phase realization sturcture is used to represent,

a) FIR systems b) IIR systems
c) both FIR and IIR systems d) all discrete time systems

4. The effect of quantization of coefficients on the frequency response is minimized in,

a) cascade realization b) parallel realization
c) direct form structure d) both a and b

5. The direct form-I and II structures of IIR system will be identical in,

a) all pole system b) all zero system
c) both a and b d) first order and second order systems.

6. The condition for symmetry of impulse response of FIR system is,

a) h(n) = h(N–1) b) h(n) = h(N+1) c) h(n) = h(N–n) d) h(n) = h(N–1–n)

7. The linear phase realization is used in FIR systems, mainly to minimize,

a) multipliers b) memory c) delays d) adders

8. Which one of the following FIR system has linear phase response?

a) y(n) = 0.4 x(n) + 0.1 x(n–1) + 0.5 x(n–2) b) y(n) = 0.3 x(n) + x(n–1) + 3.0 x(n–2)

c) y(n) = 0.5 x(n) + 0.7 x(n–1) d) y(n) = 0.6x(n) + 0.6 x(n–1)

9. The quantization error increases, when the order of the system 'N' increases in case of,

a) direct form realization b) cascade or parallel form realization
c) all IIR systems d) all FIR systems

10. The number of memory locations required to realize the system, $H(z) = \dfrac{1+z^{-2}+2z^{-3}}{1+z^{-2}+z^{-4}}$ is,

a) 8 b) 7 c) 2 d) 10

11. Number of multipliers and adders required for direct form realization of N^{th} order FIR system are,

a) N, N+1 b) N, N–1 c) N+1, N, N d) N–1, N+1

12. The realization of linear phase FIR system for odd values of 'N' needs,

a) $\frac{N}{2}$ multipliers b) $\frac{N+1}{2}$ multipliers c) N–1 multipliers d) N multipliers

Answers

1. d	3. a	5. c	7. a	9. a	11. b
2. b	4. d	6. d	8. d	10. b	12. b

IV. Answer the following questions

1. What are the various issues that are addressed by realization structures?
2. What are the basic elements used to construct the realization structures of discrete time system?
3. List the different types of structures for realization of IIR systems.
4. Draw the direct form-I structure of an N^{th} order IIR system with equal number of poles and zeros.
5. Draw the direct form-II structure of an N^{th} order IIR system with equal number of poles and zeros.
6. Explain the conversion of direct form-I structure to direct form-II structure with an example.
7. What are the difficulties in cascade realization?
8. Explain the realization of cascade structure of an IIR system.
9. Explain the realization of parallel structure of an IIR system.
10. What are the different types of structure for realization of FIR systems?
11. Draw the direct form structure of an N^{th} order FIR system.
12. What is the necessary condition for Linear phase realization of FIR system?
13. Draw the linear phase realization structure of an Nth order FIR system when 'N' is even.
14. Draw the linear phase realization structure of an Nth order FIR system when 'N' is odd.
15. Explain the realization of cascade structure of a FIR system.

V. Solve the following problems

E10.1 Obtain the direct form-I, direct form-II, cascade and parallel form realizations of the LTI system governed by the equation,

$$y(n) = -\frac{5}{4}\,y(n-1) + \frac{1}{8}\,y(n-2) + \frac{1}{16}\,y(n-3) + x(n) + 5x(n-1) + 6x(n-2)$$

E10.2 Realize the direct form-I and direct form-II of the IIR system represented by the transfer function,

$$H(z) = \frac{2(z+2)}{(z-0.1)\,(z+0.5)\,(z+0.4)}$$

E10.3 Determine the direct form-I, II, cascade and parallel realization of the following LTI system.

$$H(z) = \frac{(z^3 - 2z^2 + 2z - 1)}{(z-0.5)\,(z^2 + z - 0.5)}$$

E10.4 Realize the cascade and parallel structures of the system governed by the difference equation,

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + \frac{1}{2}x(n-1)$$

E10.5 Draw the direct form structure of the FIR systems described by the following equations,

a) $y(n) = x(n) + \frac{1}{3}x(n-1) + \frac{1}{4}\,x(n-2) + \frac{1}{5}\,x(n-3) + \frac{2}{7}\,x(n-4)$

b) $y(n) = 0.35\,x(n) + 0.3\,x(n-1) + 0.125x(n-2) - 0.25x(n-3)$
$- 0.35x(n-4) - 0.3x(n-5) - 0.125x(n-6)$

E10.6 Realize the following FIR systems with minimum number of multipliers.

a) $H(z) = 0.2 + 0.5z^{-1} + 0.3z^{-2} + 0.5z^{-3} + 0.2z^{-4}$

b) $H(z) = \left(1 + \frac{1}{8}\,z^{-1} + z^{-2}\right)\left(2 - \frac{1}{9}\,z^{-1} + 2z^{-2}\right)$

c) $y(n) = -\frac{1}{2}\,x(n) + \frac{3}{5}\,x(n-1) + \frac{3}{8}\,x(n-2) + \frac{3}{5}\,x(n-3) - \frac{1}{2}x(n-4)$

Answers

E10. 1

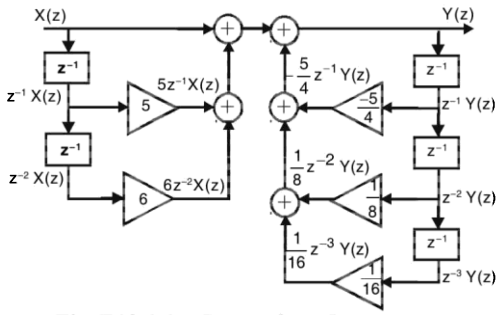


Fig E10.1.1 : Direct form-I structure.

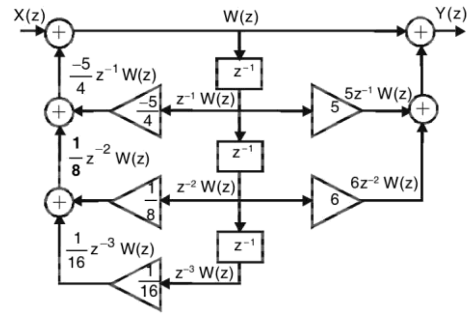


Fig E10.1.2 : Direct form-II structure.

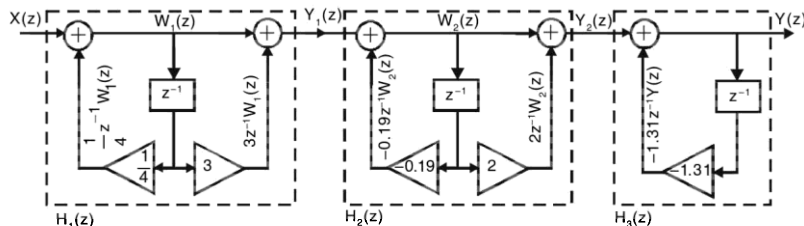


Fig E10.1.3 : Cascade structure.

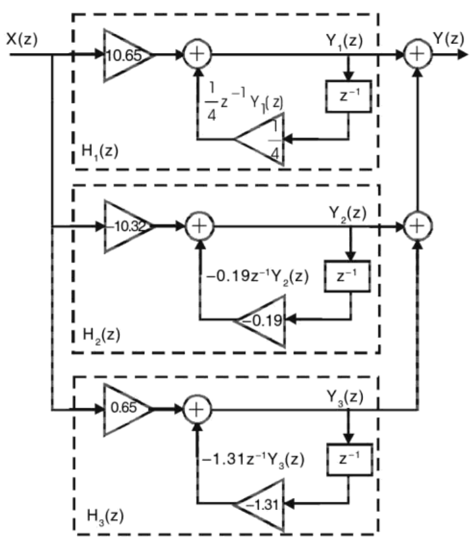


Fig E10.1.4 : Parallel structure.

E10. 2

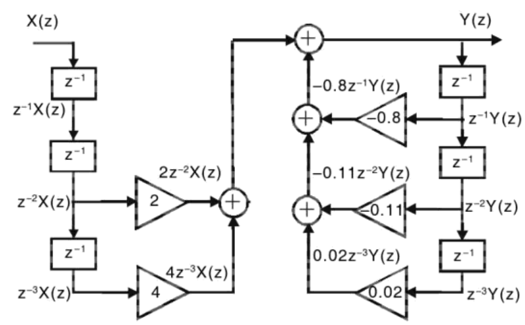


Fig E10.2.1 : Direct form-I structure.

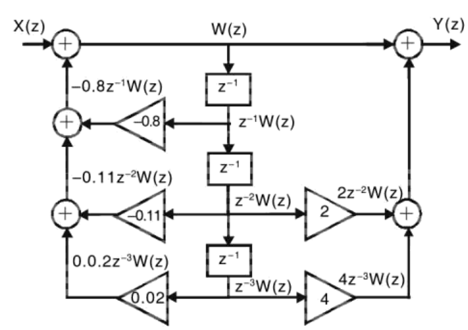


Fig E10.2.2 : Direct form-II structure.

E10.3

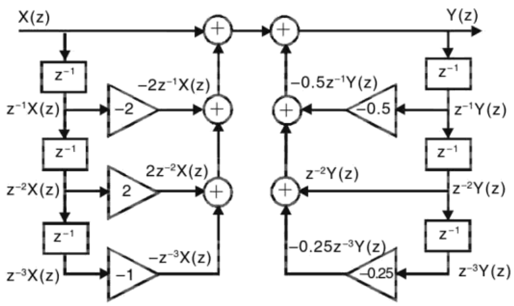


Fig E10.3.1 : *Direct form-I structure.*

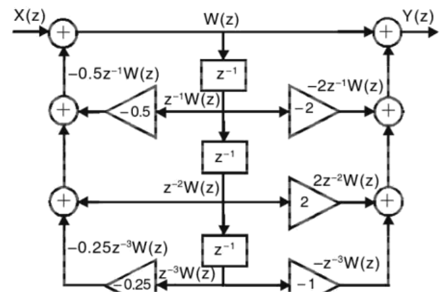


Fig E10.3.2 : *Direct form-II structure.*

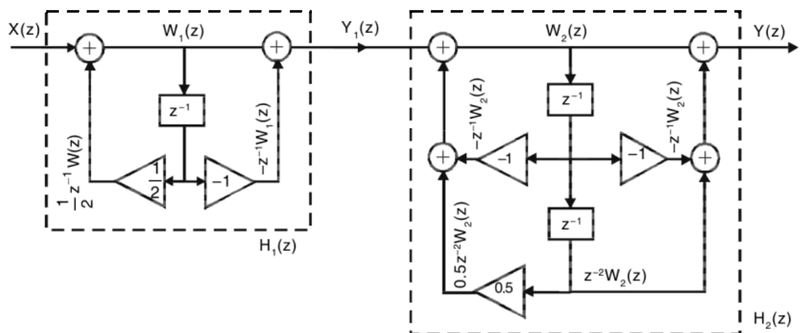


Fig E10.3.3 : *Cascade structure.*

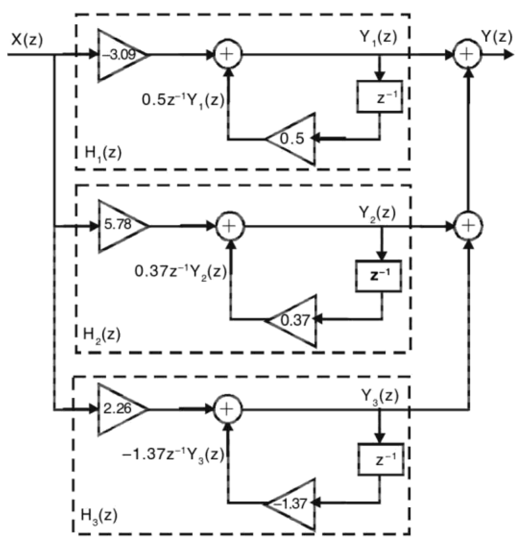


Fig E10.3.4 : *Parallel structure.*

E10.4

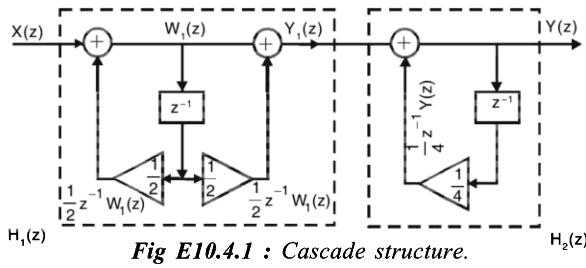


Fig E10.4.1 : Cascade structure.

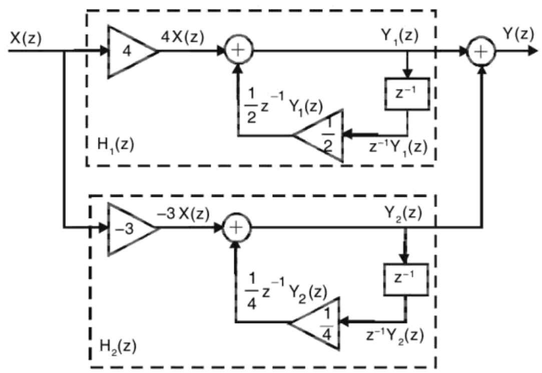


Fig E10.4.2 : Parallel structure.

E10.5 a)

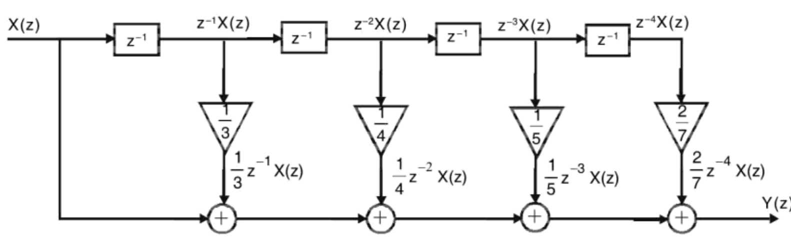


Fig E10.5a : Direct form structure.

E10.5 b)

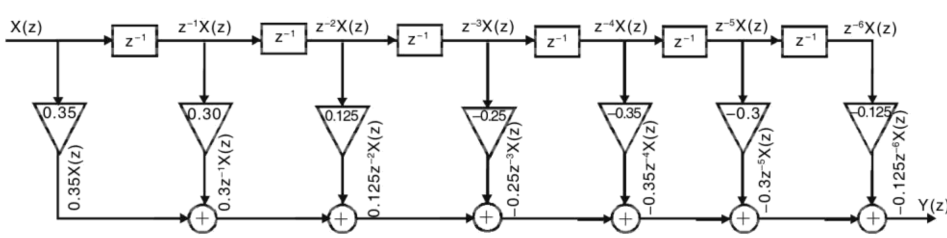


Fig E10.5b : Direct form structure.

E10.6 a)

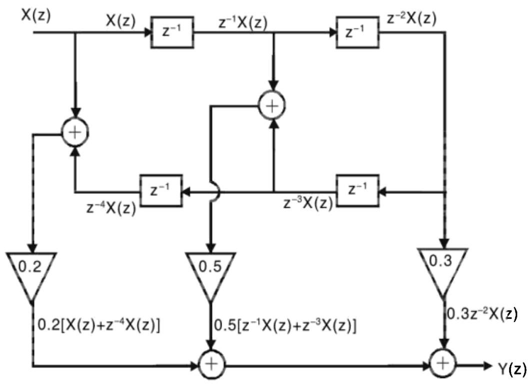


***Fig E10.6a :** Linear phase structure.*

E10.6 b)

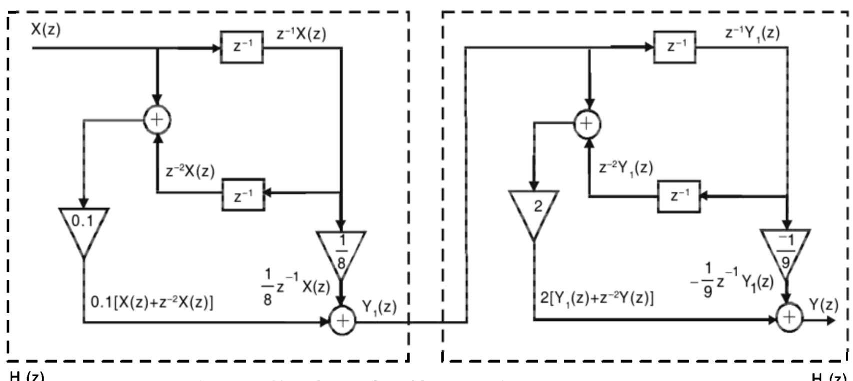


***Fig E10.6b :** Cascade of linear phase structure.*

E10.6 c)

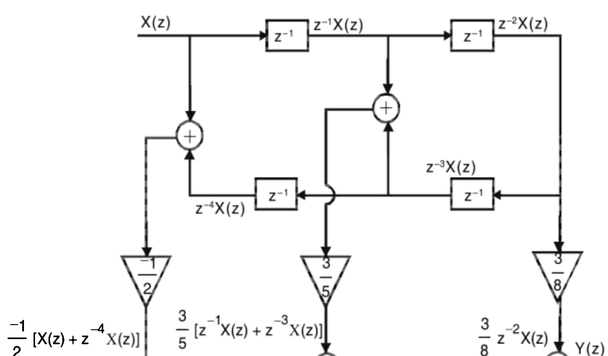


***Fig E10.6c :** Linear phase structure.*

QUESTION BANK

LINEAR TIME INVARIANT DISCRETE TIME SYSTEMS

PART-A

1.Distinguish between recursive and non-recursive systems.(N/D'15,M/J'17)
2.Convolve the following signals, x[n] ={1,3,5} and h[n] ={1,4,-1} (N/D'15)
3.Name the basic building blocks used in LTIDT system block diagram. (M/J'15)
4.Write the n^{th} order difference equation . (M/J'15)
5.Give the impulse response of a linear time invariant as h(n)=sin πn, check whether the system is stable or not. (N/D'14)
6.Interms of ROC, state the condition for an LTI discrete time system to be causal and stable. (N/D'14)
7.Find the overall impulse response h(n)= when two systems $h_1(n)=u(n)$ and $h_2(n)=\delta(n)+2\delta(n-1)$ are series. (M/J'14)
8.Using z-transform,check whether the following system is stable H(z)= $\frac{z}{z-\frac{1}{2}} + \frac{2z}{z-3}$ for ROC ½ <|z| <3 (M/J'14)

9.Define convolution sum with its equation. (N/D'13)
10.Check whether the system with system function H(Z) = $\frac{1}{1-\frac{1}{2}z-1} + \frac{1}{1-2z-1}$ with ROC |Z|<1/2 is causal or stable. (N/D'13)

11.Is the discrete time system described by the difference equation y(n)=x(-n) causal. (M/J'13)
12.If X(ω) is the DTFT of x(n),what is the DTFT of x(-n) ? (M/J'13)
13.Convolve the following two sequences x(n) ={1,1,1,1} and h(n)={3,2} (N/D'12)
14.A causal LTI system has impulse response h(n) for which the z-transform is H(z) = $\frac{1+z-1}{(1-0.5z-1)(1+0.25z-1)}$ is the system stable ? Explain (N/D'12)

15.List the advantages of the state variable representation of a system.(M/J'12)
16.Find the system function for the given difference equation y(n) =0.5 y(n-1) +x(n). .(M/J'12)

17.Find the convolution of the two sequences of x(n) ={1,1,1,1} and h(n)={2,2} (N/D'11)
18.What is the overall impulse response h(n) when two system with impulse responses $h_1(n)$ and $h_2(n)$ are connected in parallel and in series ? (N/D'11)

19.Obtain the convolution of (a). x(n) * δ(n) (b). δ(n) *[$h_1(n)$ +$h_2(n)$]. (M/J'11)
20.Write the difference equation for non recursive system. (M/J'11)
21.Convolve the following sequences (N/D'16)
x[n] ={1,2,3}

h[n]={1,1,2}
22.Given the system function H(z) $=2+3z^{-1}+4z^{-3}-5z^{-4}$. Determine the impulse response h[n]. (N/D'16)
23. What is the necessary and sufficient condition on impulse response for stability of a causal LTI system ?(M/J'17)
24. .Write the condition for stability of a DT-LTI system with respect to the positon of poles.(N/D'17)
25.Realize the difference equation y[n] = x[n] – 3x[n-1] in direct form I. (N/D'17)
26.Draw the block diagram representation of the system given its input output relationship. Y[n]=$\sum_{k=0}^{4} h(K)x(n-k)$. (M/J'18)
25.convolve the following signals. (M/J'18)
X[n]={1,2,-2} and h[n]={1,2,2}.
26.The input x[n] and output y[n] of a discrete time LTI system is given as x[n]={1,2,3,4} and y[n] = {0,1,2,3,4}. Find the impulse response h[n]. (N/D'18)
27.Given the system function $H(Z) = \frac{Z-1}{Z-2+2Z-1+4}$ .Find the difference equation representation of the system. (N/D'18)
28.Determine the Z transforms of the following two signals. Note that the Z transform for both have the same algebraic expression and differ only in the ROC. $x1[n] = \frac{1}{2}^{n} u[n]$. and (M/J'19)

$$x2[n] = -\frac{1}{2}^{n} u[-n-1]$$

29.Find the initial and final values of the function . (M/J'19)

$$X(Z) = \frac{1+Z^{-1}}{1-0.25Z^{-2}}$$

PART-B

1. (i) Obtain the cascade realization of
y[n] - $\frac{1}{4}$ y[n-1] - $\frac{1}{8}$ y[n-2] = x[n] + 3x[n-1] +2x[n-2] (M/J'11)

(ii) Obain the relationship between DTFT and Z transform

2. Determine the system function and impulse response of the LTI system (N/D'11)
y[n] - 0.5 y[n-1] +1/4 [(n-1] =x[n]
3. Find the impulse response of the difference equation . (M/J'12)
y[n] – 2y[n-2] +y[n-1] + 3y[n-3] = x[n] +2x[n-2].
4. (i) Draw the direct form –I block diagram representation for the system. (M/J'12
H(Z) $=(1+2Z^{-1} -20Z^{-2}-20Z^{-3}-5Z^{-4}+6Z^{-6}) / (1+0.5Z^{-1}-0.25Z^{-2})$

(ii)Find the input x[n] which produces the output y[n] = {3,8,14,8,3} , when passed through the system having h[n] = {1,2 3}
5.(i) Find the system function and the impulse response h[n] for a system described by the following input-output relationship y[n] = 1/3 y[n-1] +3x[n] (N/D'12)

(ii) A linear time –invariant system is characterized by the system function. Specify the ROC of H(Z) , $H(Z) = (3-4z^{-1}) / (1-3.5z^{-1}+1.5z^{-2})$ and determine h[n] for the following conditions:

(1) The system is stable (2) The system is causal (3) The system is anti-causal

6.(i) Derive the necessary and sufficient condition for BIBO stability of an LSI system. (N/D'12)

(ii) Draw the direct form ,cascade form and parallel form block diagrams of the following system function $H(Z) = \frac{1}{\left(1+\frac{1}{2}z-1\right)\left(1-\frac{1}{4}z-1\right)}$

7.(i) Obtain the impulse response of the system given by the difference equation

$y[n] - \frac{5}{6} y[n-1] + \frac{1}{6} y[n-2] = x[n]$ (M/J'13)

(ii)Determine the range of values of the parameter "a" for which the LTI system with impulse response $h[n] = a^n u[n]$ is stable.

8. Compute the response of the system y[n] = 0.7y[n-1] -0.12y[n-2]+x[n-1] + x[n-2] to the input x[n] = nu[n] . Is the system stable? (M/J'13)

9.(i) Compute convolution sum of the following sequence $x[n] = \begin{Bmatrix} 1, & 0 \le n \le 4 \\ \text{zero}, & \text{Otherwise} \end{Bmatrix}$

And $h[n] = \{\alpha^n, 0 \le n \le 6$ (N/D'13)

0, otherwise

(ii)Determine the direct form I and direct form II of the system described by the difference equation

$y[n] + \frac{1}{4} y[n-1] + \frac{1}{8} y[n-2] = x[n] + x[n-1]$

10. Determine the transfer function and the impulse response for the causal system described by the difference equation using Z transform (N/D'13)

$y[n] - \frac{1}{4} y[n-1] - \frac{3}{8} y[n-2] = - x[n] + 2x[n-1]$

11. Compute y[n] = x[n] * h[n] where x[n] = (0.5) n-2 and h[n] = u[n-2]. (N/D'14)

12. LTI discrete time system y[n] = 3/2 y[n-1] – 1/2 y[n-2] + x[n] + x[n-1] is given an input x[n] =u[n] (N/D'14)

(i) Find the transfer function of the system

(ii) Find the impulse response of the system

13.(i) Determine the impulse response and step response. (M/J'15)

y(n)+y(n-1)-2y(n-2)= x(n-1)+ 2x(n-2)

14.Find the convolution sum between x(n) ={1,4,3,2} and h(n) ={1,3,2,1} (M/J'15)

15.(i) A causal system has x(n) =δ(n) +1/4 δ(n-1) – 1/8 δ(n-2) and y(n)=δ(n)-3/4δ(n-1). Find the impulse response and output if $x(n) = (1/2)^n u(n)$ (M/J'15)

(ii) Compare recursive and non recursive systems.

16. Convolve the following signals:(N/D'15)

$x[n]=(1/2)^{n-2} u[n-2]$ h[n] =u[n+2]

17. Consider an LTI system with impulse response $h[n] = \alpha^n u(n)$ and input to the system is $x(n) = \beta^n u(n)$ with |α| & |β| < 1. Determine response y[n]. (N/D'15)

1.when$\alpha=\beta$
2.when$\alpha\neq\beta$ Using DTFT.
18. Convolve the following signals (N/D'16)
x[n]=u[n]-u[n-3] h[n]=$(0.5)^n$ u[n]
19.Determine whether the given system is stable by finding H(z) and plotting the pole-zero diagram. (N/D'16)
y[n]=2y[n-1]-0.8y[n-2]+x[n]+0.8x[n-1].
20. A causal system has input x[n] and output y[n].Find the
1.System function H(z)
2.Impulse Response h[n]
3.Frequency response H($e^{j\omega}$)
x[n]=$\boldsymbol{\delta}$[n]+1/6 $\boldsymbol{\delta}$[n-1] – 1/6 $\boldsymbol{\delta}$[n-2]
h[n]=$\boldsymbol{\delta}[n]^6$ – 2/3 $\boldsymbol{\delta}$[n-1]
21. Perform convolution to find the response of the systems $h_1(n)$ and $h_2(n)$ for the input sequences $x_1(n)$ and $x_2(n)$ respectively. (M/J'17)
1.$x_1(n)$ ={1,-1,2,3} $h_1(n)$ ={1,-2,3,-1}
2.$x_2(n)$ ={1,2,3,2} $h_2(n)$={1,2,2}
22. For a causal LTI system the input x(n) and output y(n) are related through a difference equation y(n)-$\frac{1}{6}$y(n-1)-$\frac{1}{6}$y(n-2) =x(n). Determine the frequency response H($e^{j\omega}$) and the impulse response h(n) of the system. (M/J'17)
23. Determine the steady state response for the system with impulse response h(n)=$[j\,0.5]^n$ for an input x(n)=cos(πn) u(n). (M/J'17)
24. Convolve the following sequences (M/J'18)
X[n]=a^nu[n],a<1. h[n]=u[n]. (13)
b).The system function H(z)=$\frac{z^2}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$ ROC:$|z|$>$\frac{1}{2}$. (13)
Determine the step response of the system. (M/J'18)
25. a)(i)Obtain the parallel realization of the system given by
y(n)-3y(n-1)+2y(n-2) = x(n) (6) (N/D'17)
(ii) Determine the Direct form II structure for the system given by difference equation.(7)
y[n] = 0.5 y[n-1]-0.25y[n-2]+x[n] +x[n-1].
26.Using the properties of inverse Z transform solve (5+5+3) (N/D'17)

(i) $X(z)=\log(1+az^{-1});|z|>|a|$ and $X(z)=\dfrac{az^{-1}}{\left(1-az^{-1}\right)^2}\ ;|z|>|a|$

(ii) Check whether the system function is causal or not

$$H(z)=\frac{1}{1-(1/2)z^{-1}}+\frac{1}{1-2z^{-1}}\quad ;|z|>2$$

(iii) Consider a system with impulse response $H(s)=\dfrac{e^s}{S+1};\mathrm{Re}\{s\}>-1$ check whether the system function is causal or not.
27. Let y[n] = x[n]*h[n] where x[n] =$(1/3)^n$ u[n] and h[n] = $(1/5)^n$ u[n] . Find y(z) by using the convolution property of z-transform and find y[n] by taking the inverse transform of y(z) using the partial fraction expansion method. (N/D'18)
28. A Causal DT LTI system is described by the difference equation $y[n-2]-\frac{7}{10}y[n-1]+\frac{1}{10}y[n]=x[n]$

Determine the system function H(Z).Also plot the pole-zero plot and determine whether the system is stable. (N/D’18).

29.(i) Find the inverse Z transform of (8) (M/J’19)

$X(Z) = \frac{1-\frac{1}{2}Z^{-1}}{1+\frac{3}{4}Z^{-1}+\frac{1}{8}Z^{-2}}$; $|z| > \frac{1}{2}$

(ii) Compute discrete-time Fourier transform of $x[n] = \alpha^n$ for $0 \le n \le N-1$.(5)

30.(i) Determine the Z transform and ROC of the given sequence .(5)

$$x[n] = -\frac{1}{3}^n u[n] - \frac{1}{2}^n u(-n-1)$$

(ii)Obtain the direct form I and direct form II realization of the LTI system (M/J’19)

$$y[n] = \frac{3}{32}y[n-2] - \frac{3}{8}y[n-1] + \frac{1}{64}y[n-3] + x[n] + 3x[n-1] + 2x[n-2]$$

PART-C

1.a)State and explain sampling theorem with necessary equations and illustrations. (M/J’18)

b) A discrete time system is both linear and time invariant. The output produced by this system for an impulse input is {1,2,3}.(M/J’18)

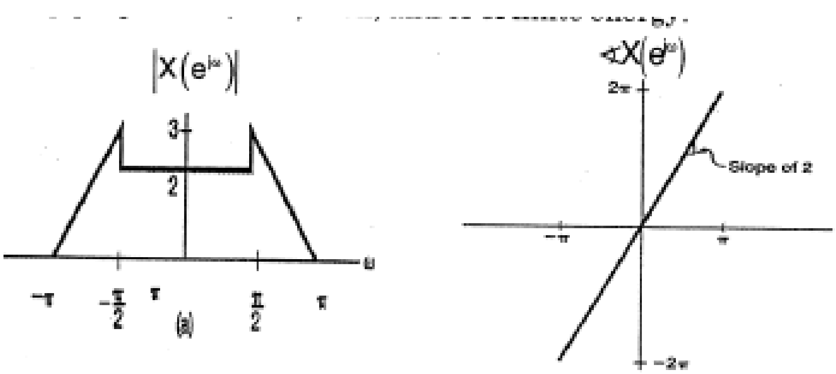
Find the output of this for the following inputs and justify your answer: (5+5+5)

i)$\delta[n-2]$.

ii)$\delta[n] - 2\delta[n-1]$

iii){1,2,3}

2. a)(i) Consider a sequence x[n] whose Fourier transform X(ω) is depicted for $-\pi \le \omega \le \pi$ in the figure below. Determine whether or not , in the domain ,x[n] is periodic, real, even, and/or of finite energy. (6) (N/D’17)



(ii)What is the transfer function and the impulse response of low pass RC circuit? (5)

(iii) Find the necessary and sufficient condition on the impulse response h[n] such that for any input x[n]

Max{|x[n]|} ≥ max {|y[n]|}, where y[n] = x[n] *h[n]. (4)

b) Analyze on recursive and non-recursive systems with an example. (15) (N/D’17)

3.a) Given the impulse response of a discrete time LTI system (N/D’18) (15)

$h[n] = [-2(1|3)n + 3(1|2)n]u[n]$

(i) Find the system function H(Z) of the system.

(ii) Find the difference equation representation of the system.

(iii) Find the step response of the system .

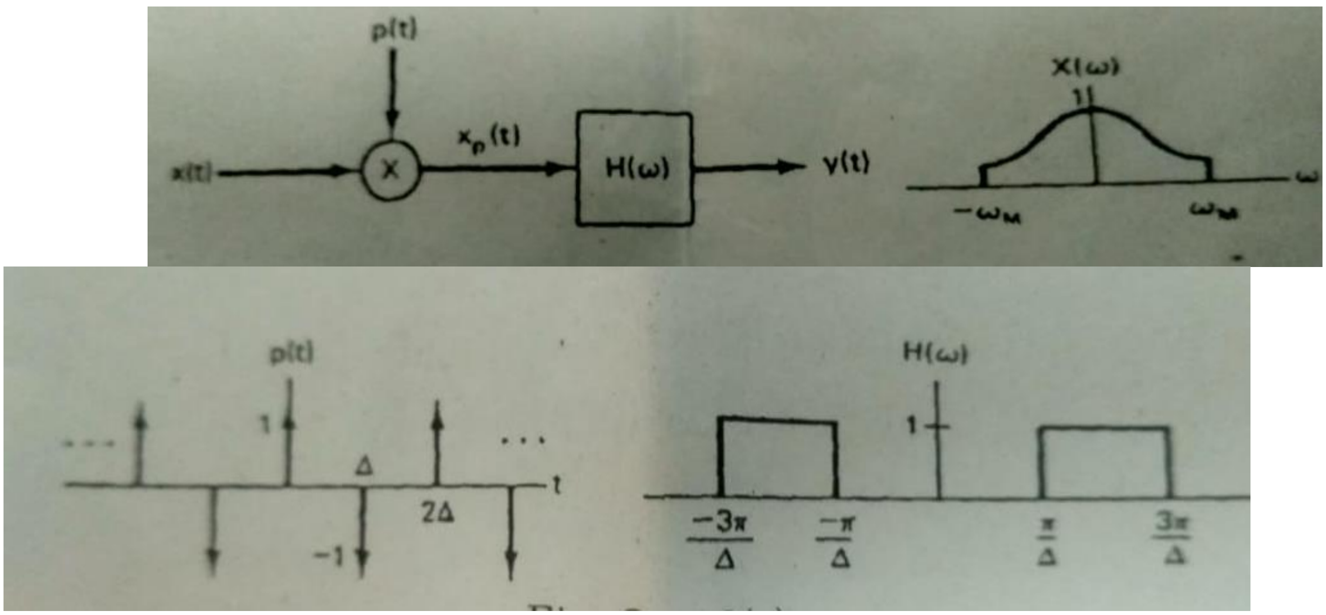
a) The input output relationship of a discrete time system is given by (N/D'18) (15)

$y[n] - \frac{1}{4}y[n-1] = x[n]$

Find the response y[n] if the Fourier transform of the input x[n] is given as

$X(e^{j\omega}) = \frac{1}{1-0.5e^{-j\omega}}$

4.a).(i) A system in which the sampling signal p(t) is an impulse train with alternating sign is given in the figure(a). The Fourier transform x(ω) of the input signal are x(t) and the Fourier transform H(ω) as indicated in the figure. (11) (M/J'19)



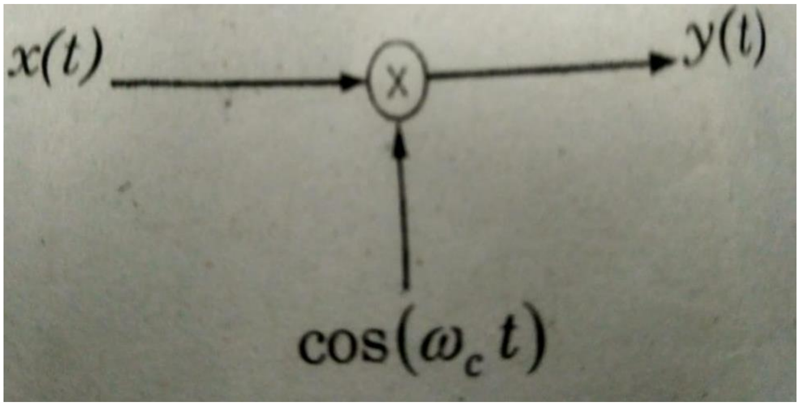
(1) For $\Delta < \pi/2\omega_M$, sketch the Fourier transform of $x_p(t)$ and y(t)

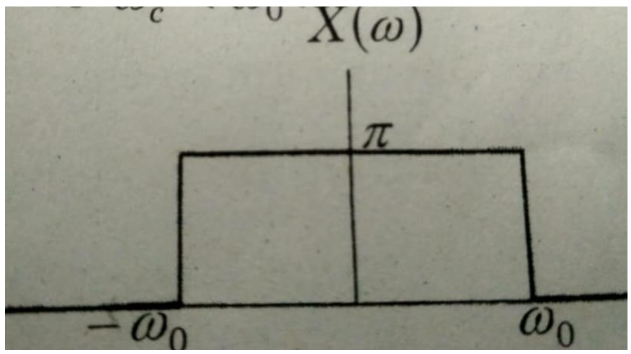
(2) For $\Delta < \pi/2\omega_M$,determine a system that will recover x(t) from $x_p(t)$.

(3) For $\Delta < \pi/2\omega_M$,determine a system that will recover x(t) from y(t).

(4) What is the maximum value of Δ in relation to ω_M for which x(t) can be recovered from either $x_p(t)$ or y(t).

(ii) Using figure (b) determine y(t) and sketch Y(ω) if X(ω) is given by figure (c) .Assume $\omega_c < \omega_0$ (4)





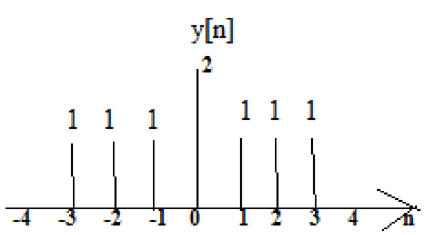
(b)(i) (1) Suppose that the signal $e^{j\omega t}$ is applied as the excitation to a linear,time-invariant system that has an impulse response h(t). By using the convolution integral, show that the resulting output is H(ω) $e^{j\omega t}$, Where H(ω) =

$$\int_{-\infty}^{\infty} h(\tau)e^{j\omega\tau}\, d\tau$$

(2).Assume that the system is characterized by a first order differential equation $\frac{dy(t)}{dt} + \alpha$ *y(t) =x(t).*

If x(t) $= e^{j\omega t}$ for all t, then y(t)= H(ω) $e^{j\omega t}$, for all t. By substituting into the differential equation, determine H(ω). (8) (M/J'19)

(ii) Consider the signal y[n].



(1) Find a signal x[n] such that Even $\{x[n]\} = y[n]$ for n ≥ 0 , and Odd $\{x[n]\} = y[n]$ for n < 0.

(2) Suppose the Even $\{w[n]\} = y[n]$ for all n .Also assume that w[n] =0 for n < 0. Find w[n].