UNIT-5

# Applications of partial differential Equations

## * One Dimensional Wave Equations :-

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

where $a^2 = \frac{T}{m}$

$$a^2 = \frac{\text{Tension}}{\text{Mass per unit length of the String}}$$

## * Three possible Solutions :-

(i) $y = (Ae^{px} + Be^{-px})(ce^{pat} + De^{-pat})$

(ii) $\boxed{y = (A\cos px + B\sin px)(C\cos pat + D\sin pat)}$

(iii) $y = (Ax+B)(ct+D)$

## * Type : 1 ZERO VELOCITY (string)

(i) $y=0$ when $x=0$

(ii) $y=0$ when $x=l$ (or) $2l$

(iii) $\frac{\partial y}{\partial t} = 0$ when $t=0$ (zero velocity)

(iv) $y = f(x)$ when $t=0$

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* Working Rule :- (Type : 1 ZERO VELOCITY)

The one Dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The Conditions are

(i) $y=0$ when $x=0$

(ii) $y=0$ when $x=l$

(iii) $\frac{\partial y}{\partial t} = 0$ when $t=0$ (zero velocity)

(iv) $y=f(x)$ when $t=0$

The suitable solution is

$$\boxed{y = (A \cos px + B \sin px)(C \cos pat + D \sin pat)} \rightarrow (1)$$

Step : 1 Applying Condition (i) in eqn. (1)

$y=0$ & $x=0$

$$0 = (A \cos 0 + B \sin 0)(C \cos pat + D \sin pat)$$

$$0 = (A+0)(C \cos pat + D \sin pat)$$

$$0 = A(C \cos pat + D \sin pat)$$

If $(C \cos pat + D \sin pat) \neq 0$,

$$\therefore \boxed{A=0}$$

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Sub. $A=0$ in (1)

$$y = (0 + B\sin px)(C\cos pat + D\sin pat)$$

$$\boxed{y = B\sin px\,(C\cos pat + D\sin pat)} \rightarrow (2)$$

<u>Step : 2</u> Applying Condition (ii) in (2)

$x=0$ $\underline{y=0}$ & $\underline{x=l}$

$$0 = B\sin pl\,(C\cos pat + D\sin pat)$$

If $B \neq 0$, $(C\cos pat + D\sin pat) \neq 0$

$\therefore \sin pl = 0$ $[\because \sin n\pi = 0$

$pl = \sin^{-1}(0)$ $n\pi = \sin^{-1}(0)]$

$pl = n\pi$

$$\boxed{p = \frac{n\pi}{l}}$$

Sub. $p = \frac{n\pi}{l}$ in (2)

$$\boxed{y = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right]}$$

$\hookrightarrow (3)$

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Step : 3

Before Applying Condition (iii)

Partially differentiate (3) w.r.to 't'

$$\frac{\partial y}{\partial t} = B \sin\left(\frac{n\pi x}{l}\right)\left[-c\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi at}{l}\right) + D\left(\frac{n\pi a}{l}\right)\cos\left(\frac{n\pi at}{l}\right)\right]$$

Now Applying Condition (iii) in above eqn.

$$\underline{\frac{\partial y}{\partial t} = 0} \quad \& \quad \underline{t = 0}$$

$$0 = B \sin\left(\frac{n\pi x}{l}\right)\left[0 + D\left(\frac{n\pi a}{l}\right)\cos 0\right]$$

$$0 = BD\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore B \neq 0, \quad \sin\left(\frac{n\pi x}{l}\right) \neq 0 \quad \& \quad \left(\frac{n\pi a}{l}\right) \neq 0$$

$$\therefore \boxed{D = 0}$$

Sub. $\underline{D = 0}$ in (3)

$$y = B \sin\left(\frac{n\pi x}{l}\right)\left[c \cos\left(\frac{n\pi at}{l}\right) + 0\right]$$

$$y = Bc \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

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put $BC = b_n$

$$y = b_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right)$$

The most general solution is

$$\boxed{y = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right)} \to (4)$$

step: 4

Applying condition (iv) in (4)

$y = f(x)$ (depends upons your problem)

&

$t = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

(Here R.H.S represent Half range Sine Series)

Then we have to find $b_n$ and substitute $b_n$ value in ④.

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## Type: 2 Non-zero Velocity - string

(i) $y=0$ when $x=0$

(ii) $y=0$ when $x=l$ or $2l$

(iii) $y=0$ when $t=0$

(iv) $\frac{\partial y}{\partial t}=f(x)$ when $t=0$ (Non-zero Velocity)

* **Working Rule** :- (Non-zero Velocity)

One Dimensional wave eqn. is

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$$

The Conditions are

(i) $y=0$ when $x=0$

(ii) $y=0$ when $x=l$

(iii) $y=0$ when $t=0$

(iv) $\frac{\partial y}{\partial t}=f(x)$ when $t=0$ (Non-zero Velocity)

The suitable solution is

$$\boxed{y=(A\cos px+B\sin px)(C\cos pat+D\sin pat)} \rightarrow (1)$$

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Step : 1

Applying Condition (i) in eqn. (1)

$\underline{y=0}$ & $\underline{x=0}$

$$0 = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$0 = (A+0)(C\cos pat + D\sin pat)$$

$$0 = A(C\cos pat + D\sin pat)$$

Here $(C\cos pat + D\sin pat) \neq 0$

$$\therefore \boxed{A=0}$$

put $\underline{A=0}$ in (1)

$$y = (0 + B\sin px)(C\cos pat + D\sin pat)$$

$$\boxed{y = B\sin px\,(C\cos pat + D\sin pat)}$$

$\hookrightarrow$ (2)

Step : 2

Applying Condition (ii) in eqn. (2)

$\underline{y=0}$ & $\underline{x=l}$

$$0 = B\sin pl\,(C\cos pat + D\sin pat)$$

Here $B \neq 0$, $(C\cos pat + D\sin pat) \neq 0$

$$\therefore \sin pl = 0$$

$$pl = \sin^{-1}(0)$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

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Sub. $P = \frac{n\pi}{l}$ in eqn. (2)

$$\boxed{y = B \sin\left(\frac{n\pi x}{l}\right)\left[C \cos\left(\frac{n\pi at}{l}\right) + D \sin\left(\frac{n\pi at}{l}\right)\right]} \rightarrow (3)$$

<u>step: 3</u>

Applying Condition (iii) in eqn. (3)

$\underline{y=0}$ & $\underline{t=0}$

$$0 = B \sin\left(\frac{n\pi x}{l}\right)\left[C \cos 0 + D \sin 0\right]$$

$$0 = B \sin\left(\frac{n\pi x}{l}\right)(C+0)$$

$$0 = BC \sin\left(\frac{n\pi x}{l}\right)$$

Here $B \neq 0$ & $\sin\left(\frac{n\pi x}{l}\right) \neq 0$

$\therefore \boxed{C=0}$

Sub. $\underline{C=0}$ in (3)

$$y = B \sin\left(\frac{n\pi x}{l}\right)\left[0 + D \sin\left(\frac{n\pi at}{l}\right)\right]$$

$$y = BD \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right)$$

put $\boxed{BD = C_n}$

$$y = C_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right)$$

The most general solution is

$$\boxed{y = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right)} \rightarrow (4)$$

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Step : 4

Before applying Condition (iv) partially differentiate eqn. (4) w.r. to 't'

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{\ell}\right)\left[\left(\frac{n\pi a}{\ell}\right)\cos\left(\frac{n\pi a t}{\ell}\right)\right]$$

Now we can sub. Condition (iv) in above eqn.

$$\frac{\partial y}{\partial t} = f(x) \quad \& \quad t = 0$$

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{\ell}\right)\left[\left(\frac{n\pi a}{\ell}\right)\cos 0\right]$$

$(\because \cos 0 = 1$

$$f(x) = \sum_{n=1}^{\infty} C_n\left(\frac{n\pi a}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

Here put $\boxed{C_n\left(\frac{n\pi a}{\ell}\right) = b_n}$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

(Now R.H.S of eqn. Converted as an Half Range Sine Series)

Next find $b_n$, based on $\boxed{b_n}$ finally we will get $\boxed{C_n}$, sub. $\boxed{C_n}$ in eqn. (4)

— x —

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# UNIT – V

## APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS

### ONE DIMENSIONAL WAVE EQUATION:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

where $a^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass per unit length of the string}}$

### THREE POSSIBLE SOLUTION:

$$y = (Ae^{px} + Be^{-px})(Ce^{pat} + De^{-pat})$$

$$y = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$$

$$y = (Ax + B)(Ct + D)$$

### TYPE: I ZERO VELOCITY [Vibrating string]

i) $y = 0$, when $x = 0$

ii) $y = 0$, when $x = l$ or $2l$

iii) $\frac{\partial y}{\partial t} = 0$, when $t = 0$ (zero velocity)

(v) $y = f(x)$, when $t = 0$

### WORKING RULE:

Step: 1 $A = 0$

Step: 2 $p = \frac{n\pi}{l}$

Step: 3 $D = 0$

Step: 4 $y = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right)$

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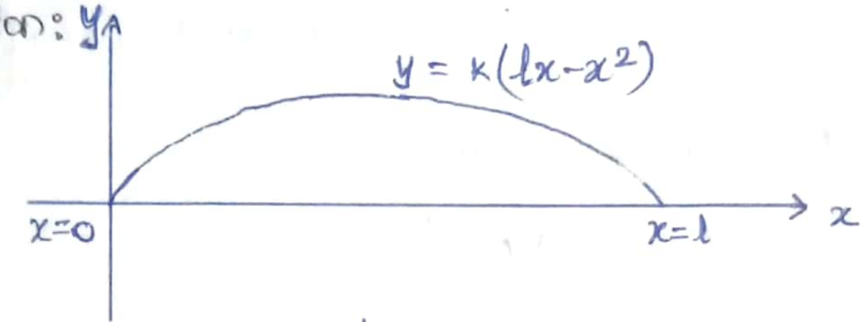
Problem:1

A tightly stretched string has its end fixed $x=0$ and $x=l$ at time $t=0$. the string is given a shape defined by $f(x) = k(lx-x^2)$, where $k$ is a constant and then released from rest. Find displacement of any point $x$ of the string at any time $t>0$.

*solution:




The one dimensional wave eqn. is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

∴ The solution is

$$y = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \longrightarrow ①$$

The conditions are

i) $y=0$ when $x=0$

ii) $y=0$ when $x=l$

iii) $\frac{\partial y}{\partial t}=0$ when $t=0$

iv) $y=f(x)=k(lx-x^2)$ when $t=0$

Step:1 Applying condition ① in ①

$$0 = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$0 = A(C\cos pat + D\sin pat)$$

If $[C\cos pat + D\sin pat] \neq 0$ ∴ $A=0$

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Sub. $A=0$ in ①

$$y = (0 + B\sin px)(c\cos pat + D\sin pat)$$

$$y = B\sin px(c\cos pat + D\sin pat) \longrightarrow ②$$

Step: 2 Applying cond. (ii) in ②

$$0 = B\sin pl\,(c\cos pat + D\sin pat)$$

If $c\cos pat + D\sin pat \neq 0$, $B \neq 0$

$$\therefore \sin pl = 0 \qquad \text{WKT } \sin n\pi = 0$$

$$pl = \sin^{-1}(0) \qquad n\pi = \sin^{-1}(0)$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Sub. $p = \frac{n\pi}{l}$ in ②

$$y = B\sin\left(\frac{n\pi x}{l}\right)\left(c\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \longrightarrow ③$$

Step: 3 Partial diff. ③ w.r. to 't'

$$\frac{\partial y}{\partial t} = B\sin\left(\frac{n\pi x}{l}\right)\left[-c\sin\left(\frac{n\pi at}{l}\right) + D\left(\cos\left(\frac{n\pi at}{l}\right)\right)\right]\left(\frac{n\pi a}{l}\right)$$

Applying cond. (iii) in above eqn.

$$\frac{\partial y}{\partial t} = 0 \quad \text{when } t=0$$

$$0 = B\sin\left(\frac{n\pi x}{l}\right)\left[D\cos\left(\frac{n\pi a(0)}{l}\right)\right]\left(\frac{n\pi a}{l}\right)$$

$$0 = BD\sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi a}{l}\right)$$

$$0 = BD\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

If $B \neq 0$, $\sin\left(\frac{n\pi x}{l}\right) \neq 0$, $\frac{n\pi a}{l} \neq 0$

$$\therefore D = 0$$

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Sub. $D=0$ in (8)

$$y = B\sin\left(\frac{n\pi x}{l}\right) C\cos\left(\frac{n\pi at}{l}\right)$$

$$y = BC\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

Put $BC = b_n$

$$y = b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

The most general solution is

$$y = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \longrightarrow (7)$$

Step: 4 Applying cond. (iv) in (7)

$y = f(x) = K(lx - x^2)$ when $t=0$

$$K(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos 0$$

$$f(x) = K(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Here R.H.S represents Half Range Sine Series.

To find $b_n$:

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l K(lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{2K}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

By using Bernoulli's theorem,

$$\int u\,v\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

$u = lx - x^2$ $\qquad v = \sin\left(\frac{n\pi x}{l}\right)$ $\qquad v_2 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$

$u' = l - 2x$ $\qquad v_1 = \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}$

$u'' = -2$ $\qquad v_3 = \frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(n\pi/l\right)^3}$

$u''' = 0$

$$b_n = \frac{2k}{l}\left[(lx-x^2)\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right] + (l-2x)\left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right] + \right.$$

$$\left. - \frac{2\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l$$

$$b_n = \frac{2k}{l}\left\{\left[0+0-\frac{2\cos n\pi}{\left(\frac{n\pi}{l}\right)^3}\right] - \left[0+0-\frac{2\cos 0}{\left(\frac{n\pi}{l}\right)^3}\right]\right\}$$

$$b_n = \frac{2k}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right] \qquad \therefore \cos n\pi = (-1)^n$$

$$\cos 0 = 1$$

$$b_n = \frac{2k}{l}\left(\frac{2l^3}{n^3\pi^3}\right)\left[1-(-1)^n\right]$$

$$b_n = \frac{4kl^2}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$b_n = \begin{cases} \frac{8kl^2}{n^3\pi^3} & \text{if } n = \text{odd} \\ 0 & \text{if } n = \text{even} \end{cases}$$

Sub. $b_n$ in (4)

$$y = \sum_{n=\text{odd}}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\left(\frac{n\pi x}{l}\right) \cdot \cos\left(\frac{n\pi at}{l}\right)$$

$$y = \frac{8kl^2}{\pi^3} \sum_{n=\text{odd}}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right)$$

Problem: 2

A tightly stretched string of length '$l$' has its end fastened at $x=0$ and $x=l$. The mid-point of the string is then taken to height '$b$ (or) $h$' and released from rest in that position. Find the lateral displacement of a point of the string at time '$t$' from the instant of release.

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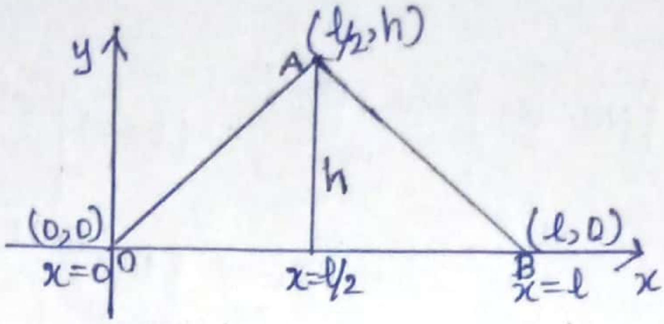


The One Dimensional wave eqn is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The solution is

$$y = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \longrightarrow (1)$$

The conditions are

i) $y = 0$ when $x = 0$

(ii) $y = 0$ when $x = l$

iii) $\frac{\partial y}{\partial t} = 0$ when $t = 0$

iv) $y = f(x)$ when $t = 0$

The Equation of OA is

$(0,0)$ $(l/2, h)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 0}{h - 0} = \frac{x - 0}{\frac{l}{2} - 0}$$

$$\frac{y}{h} = \frac{2x}{l}$$

$$y = \frac{2hx}{l} \; ; \quad 0 < x < \frac{l}{2}$$

The Equation of AB is

$(\frac{l}{2}, h)$ $(l, 0)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

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$$\frac{y-h}{0-h} = \frac{x-l/2}{l-l/2}$$

$$\frac{y-h}{-h} = \frac{\frac{2x-l}{2}}{l/2}$$

$$\frac{y-h}{-h} = \frac{2x-l}{l}$$

$$y-h = -h\left(\frac{2x-l}{l}\right) = \frac{-2hx+hl}{l}$$

$$y = h + \frac{hl-2hx}{l}$$

$$y = \frac{hl+hl-2hx}{l} = \frac{2hl-2hx}{l}$$

$$y = \frac{2h}{l}(l-x) ; \quad \frac{l}{2} < x < l$$

$$\therefore \quad y = f(x) = \begin{cases} \frac{2hx}{l} & ; \quad 0 < x < l/2 \\ \frac{2h}{l}(l-x) & ; \quad l/2 < x < l \end{cases}$$

Step:1 Applying cond. (i) in (1)

$y=0$ when $x=0$

$$0 = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$0 = A(C\cos pat + D\sin pat)$$

If $(C\cos pat + D\sin pat) \neq 0$

$\therefore A = 0$

Sub. $A=0$ in (1)

$$y = B\sin px\,(C\cos pat + D\sin pat) \longrightarrow (2)$$

Step:2 Applying cond. (ii) in (2)

$y=0$ when $x=l$

$$0 = B\sin pl\,(C\cos pat + D\sin pat)$$

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If $(C\cos pat + D\sin pat) \neq 0$ $\quad B \neq 0$

then $\quad \sin pl = 0$

$pl = \sin^{-1}(0) = n\pi$

$p = \frac{n\pi}{l}$

Sub. $p = \frac{n\pi}{l}$ in (2)

$$y = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right] \rightarrow (3)$$

Step: 3

Partial diff. (3) w.r.t. to 't'

$$\frac{\partial y}{\partial t} = B\sin\left(\frac{n\pi x}{l}\right)\left[-C\sin\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right) + D\cos\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right)\right]$$

$$\frac{\partial y}{\partial t} = B\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\left[-C\sin\left(\frac{n\pi at}{l}\right) + D\cos\left(\frac{n\pi at}{l}\right)\right]$$

Applying cond. (iii) in above eq.

$\frac{\partial y}{\partial t} = 0$ when $t = 0$

$$0 = B\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\left[-C\sin(0) + D\cos 0\right]$$

$$0 = BD\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

If $B \neq 0$, $\frac{n\pi a}{l} \neq 0$, $\sin\left(\frac{n\pi x}{l}\right) \neq 0$

then, $\therefore D = 0$

Sub. $D = 0$ in (3)

$$y = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{l}\right) + 0\right]$$

$$y = BC\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

Put $BC = b_n$

$$y = b_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

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The most general solution is

$$y = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right) \longrightarrow (4)$$

Step: 4 Applying cond. (iv) in (4)

$y = f(x)$ when $t=0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Here R.H.S represents H.R.S.S.

To find $b_n$:

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{2}{l} \left\{ \int_0^{l/2} \frac{2hx}{l} \sin\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^{l} \frac{2h}{l}(l-x) \sin\left(\frac{n\pi x}{l}\right) dx \right\}$$

$$b_n = \frac{4h}{l^2} \left\{ \int_0^{l/2} x \sin\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^{l} (l-x) \sin\left(\frac{n\pi x}{l}\right) dx \right\}$$

By using Bernoulli's theorem,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

$u = x$ $\qquad u = l-x$ $\qquad v = \sin\left(\frac{n\pi x}{l}\right)$

$u' = 1$ $\qquad u' = -1$ $\qquad v_1 = \frac{-\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}$

$u'' = 0$ $\qquad u'' = 0$

$$v_2 = -\frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}$$

$$b_n = \frac{4h}{l^2} \left\{ \left[ -x \frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} + \frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} \right]_0^{l/2} + \left[ -\frac{(l-x)\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - \frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2} \right]_{l/2}^{l} \right\}$$

$$b_n = \frac{4h}{l^2}\left\{\left[\frac{-l/2\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2} + 0 + 0\right] + \left[\frac{+l/2\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)} + \frac{\sin\left(n\pi/2\right)}{\left(\frac{n\pi}{l}\right)^2} - 0 - 0\right]\right\}$$

$$b_n = \frac{4h}{l^2}\left\{\cancel{\frac{-l/2\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)}} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2} + \cancel{\frac{l/2\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)}} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2}\right\}$$

$$b_n = \frac{4h}{\cancel{l^2}}\left[2\frac{\cancel{l^2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$b_n = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \begin{cases}\frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right), & \text{if } n \text{ is odd}\\ 0, & \text{if } n \text{ is even}\end{cases}$$

Sub. $b_n$ in (4)

$$y = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi a t}{l}\right)$$

$$y = \sum_{n=odd}^{\infty} \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi a t}{l}\right)$$

$$y = \frac{8h}{\pi^2}\sum_{n=odd}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right).\sin\left(\frac{n\pi x}{l}\right).\cos\left(\frac{n\pi a t}{l}\right).$$

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Problem: 3

A string is stretched and fastened to two points $x=0$ and $x=l$ apart. Motion is started by displacing sinusoidal $y = y_0 \sin\left(\frac{\pi x}{l}\right)$ arc of length $y_0$ released at time $t=0$. Find the displacement of any point on the string at a distance $x$ from one end at time $t$.

*Solution:

The One Dimensional wave eqn. is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The solution is

$$y = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \longrightarrow (1)$$

The conditions are

i) $y=0$ when $x=0$

ii) $y=0$ when $x=l$

iii) $\frac{\partial y}{\partial t}=0$ when $t=0$

iv) $y = f(x) = y_0 \sin\left(\frac{\pi x}{l}\right)$, when $t=0$

Step: 1 Applying cond (i) in (1) $y=0$ when $x=0$

$$0 = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$0 = A(C\cos pat + D\sin pat)$$

If $(C\cos pat + D\sin pat) \neq 0$,

then $\therefore A = 0$

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Sub. $A=0$ in ①

$$y = B\sin px\left[C\cos pat + D\sin pat\right] \longrightarrow ②$$

Step:2 Applying cond. (ii) in ②

$y=0$ when $x=l$

$$0 = B\sin pl\left[C\cos pat + D\sin pat\right]$$

If $B \neq 0$, $C\cos pat + D\sin pat \neq 0$

$\therefore \sin pl = 0$

$pl = \sin^{-1}(0) = n\pi$

$p = \frac{n\pi}{l}$

Sub. $p = \frac{n\pi}{l}$ in ②

$$y = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right] \longrightarrow ③$$

Step:3

Partial Diff. ③ w.r.t. to 't'

$$\frac{\partial y}{\partial t} = B\sin\left(\frac{n\pi x}{l}\right)\left[-C\sin\left(\frac{n\pi at}{l}\right) + D\cos\left(\frac{n\pi at}{l}\right)\right]\cdot\left(\frac{n\pi a}{l}\right).$$

$$\frac{\partial y}{\partial t} = B\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\left[-C\sin\left(\frac{n\pi at}{l}\right) + D\cos\left(\frac{n\pi at}{l}\right)\right]$$

Applying cond. (iii) in above eqn.

$\frac{\partial y}{\partial t} = 0$ when $t=0$

$$0 = B\sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi a}{l}\right)\left[-C\sin 0 + D\cos 0\right]$$

$$0 = BD\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

If $B \neq 0$, $\frac{n\pi a}{l} \neq 0$, $\sin\left(\frac{n\pi x}{l}\right) \neq 0$

$\therefore D = 0$

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Sub. $D=0$ in (8)

$$y = B\sin\left(\frac{n\pi x}{l}\right)\cdot C\cos\left(\frac{n\pi at}{l}\right)$$

$$y = BC\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

Put $BC = b_n$

$$y = b_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

The most general eqn. is

$$y = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \longrightarrow (4)$$

Step: 4 Applying cond. (IV) in (4)

$y = f(x)$ when $t=0$

$$y_0\sin\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)\cos 0$$

$$y_0\sin\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$$

$$y_0\sin\left(\frac{\pi x}{l}\right) = b_1\sin\left(\frac{\pi x}{l}\right) + b_2\sin\left(\frac{2\pi x}{l}\right) + \cdots$$

Equating $\sin\left(\frac{\pi x}{l}\right) \Rightarrow b_1 = y_0$

Equating $\sin\left(\frac{2\pi x}{l}\right), \sin\left(\frac{3\pi x}{l}\right)\cdots \Rightarrow b_2 = b_3 = b_4 \cdots = 0$

From (4) $$y = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

$$y = b_1\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi at}{l}\right) + b_2\sin\left(\frac{2\pi x}{l}\right)\cos\left(\frac{2\pi at}{l}\right) + \cdots$$

$$\therefore\; y = y_0\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi at}{l}\right).$$

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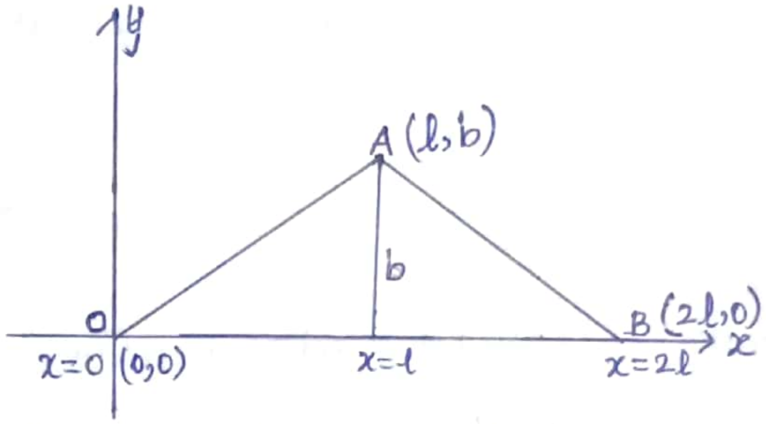
Problem:4

A string of length $2l$ is fastened at both ends. The midpoint of the string is taken to a height 'b' and then released from rest in that position. Find the displacement.

*Solution:




The One Dimensional wave eqn. is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The suitable solution is

$$y = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \longrightarrow (1)$$

The conditions are

i) $y = 0$ when $x = 0$

ii) $y = 0$ when $x = 2l$

iii) $\frac{\partial y}{\partial t} = 0$ when $t = 0$

iv) $y = f(x)$ when $t = 0$

The equation of OA is
$(0,0)$ $(l,b)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 0}{b - 0} = \frac{x - 0}{l - 0}$$

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$$\frac{y}{b} = \frac{x}{l}$$

$$y = \frac{bx}{l} \quad ; \quad 0 < x < l$$

The equation of AB is

$(l, b) \quad (2l, 0)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \Rightarrow \frac{y - b}{0 - b} = \frac{x - l}{2l - l}$$

$$\frac{y - b}{-b} = \frac{x - l}{l}$$

$$y - b = -b\left(\frac{x - l}{l}\right) = \frac{-bx + bl}{l}$$

$$y = b - \left(\frac{bx + bl}{bl}\right) = \frac{bl - bx + bl}{bl}$$

$$y = b - \frac{bx + bl}{l} = \frac{bl - bx + bl}{l}$$

$$y = \frac{2bl - bx}{l}$$

$$y = \frac{b}{l}(2l - x) \quad ; \quad l < x < 2l$$

$$\therefore \quad y = f(x) = \begin{cases} \frac{xb}{l} & ; \quad 0 < x < l \\ \frac{b}{l}(2l - x) & ; \quad l < x < 2l \end{cases}$$

Step: I Applying cond. 1 in 1

$y = 0$ when $x = 0$

$$0 = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$0 = A(C\cos pat + D\sin pat)$$

If $(C\cos pat + D\sin pat) \neq 0$

$$\therefore A = 0$$

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sub $A=0$ in (1)

$$y = B\sin px\,(C\cos pat + D\sin pat) \longrightarrow (2)$$

Step: 2 Applying cond. (ii) in (2)

$$y=0 \quad \text{when} \quad x=2l$$

$$0 = B\sin 2pl\,(C\cos pat + D\sin pat)$$

If $B \neq 0$, $(C\cos pat + D\sin pat) \neq 0$

$$\sin 2pl = 0$$

$$2pl = \sin^{-1}(0) = n\pi$$

$$p = \frac{n\pi}{2l}$$

Sub. $p = \frac{n\pi}{2l}$ in (2)

$$y = B\sin\left(\frac{n\pi x}{2l}\right)\left[C\cos\left(\frac{n\pi at}{2l}\right) + D\sin\left(\frac{n\pi at}{2l}\right)\right] \longrightarrow (3)$$

Step: 3 Partial Diff. (3) w.r.t. to 't'

$$\frac{\partial y}{\partial t} = B\sin\left(\frac{n\pi x}{2l}\right)\left[-C\sin\left(\frac{n\pi at}{2l}\right)\left(\frac{n\pi a}{2l}\right) + D\cos\left(\frac{n\pi at}{2l}\right)\left(\frac{n\pi a}{2l}\right)\right]$$

$$\frac{\partial y}{\partial t} = B\left(\frac{n\pi a}{2l}\right)\sin\left(\frac{n\pi x}{2l}\right)\left[-C\sin\left(\frac{n\pi at}{2l}\right) + D\cos\left(\frac{n\pi at}{2l}\right)\right]$$

Applying cond. (iii) in above eqn.

$$\frac{\partial y}{\partial t} = 0 \quad \text{when} \quad t=0$$

$$0 = B\left(\frac{n\pi a}{2l}\right)\sin\left(\frac{n\pi x}{2l}\right)\left[-C\sin 0 + D\cos 0\right]$$

$$0 = BD\left(\frac{n\pi a}{2l}\right)\sin\left(\frac{n\pi x}{2l}\right)$$

If $B \neq 0$, $\left(\frac{n\pi a}{2l}\right) \neq 0$, $\sin\left(\frac{n\pi x}{2l}\right) \neq 0$

$$\therefore D = 0$$

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Sub. $D=0$ in (3)

$$y = B \sin\left(\frac{n\pi x}{2l}\right) \cdot c \cos\left(\frac{n\pi at}{2l}\right)$$

$$y = BC \sin\left(\frac{n\pi x}{2l}\right) \cos\left(\frac{n\pi at}{2l}\right)$$

put $BC = b_n$

$$\Rightarrow y = b_n \sin\left(\frac{n\pi x}{2l}\right) \cos\left(\frac{n\pi at}{2l}\right)$$

The most general solution is

$$y = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2l}\right) \cos\left(\frac{n\pi at}{2l}\right) \longrightarrow (4)$$

Step: 4 Applying cond. (iv) in (4)

$y = f(x)$, when $t=0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2l}\right) \cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2l}\right)$$

Here R.H.S represents H.R.S.S.

To find $b_n$:

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Here $l = 2l$

$$b_n = \frac{\cancel{2}}{\cancel{2}l}\int_0^{2l} f(x) \sin\left(\frac{n\pi x}{2l}\right) dx$$

$$b_n = \frac{1}{l}\left\{\int_0^l \frac{bx}{l} \sin\left(\frac{n\pi x}{2l}\right) dx + \int_l^{2l} \frac{b}{l}(2l-x) \sin\left(\frac{n\pi x}{2l}\right) dx\right\}$$

$$b_n = \frac{b}{l^2}\left\{\int_0^l x \sin\left(\frac{n\pi x}{2l}\right) dx + \int_l^{2l} (2l-x) \sin\left(\frac{n\pi x}{2l}\right) dx\right\}$$

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By using Bernoulli's theorem,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

$u = x$ $\quad v = \sin\left(\frac{n\pi x}{2l}\right)$ $\quad u = 2l - x$

$u' = 1$ $\quad v_1 = \frac{-\cos\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n\pi}{2l}\right)}$ $\quad u' = -1$

$u'' = 0$ $\quad u'' = 0$

$$v_2 = \frac{-\sin\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n\pi}{2l}\right)^2}$$

$$b_n = \frac{b}{l^2}\left\{\left[\frac{-x\cos\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n\pi}{2l}\right)} + \frac{\sin\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n\pi}{2l}\right)^2}\right]_0^l + \left[\frac{-(2l-x)\cos\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n\pi}{2l}\right)} - \frac{\sin\left(\frac{n\pi x}{2l}\right)}{\left(\frac{n\pi}{2l}\right)^2}\right]_l^{2l}\right\}$$

$$b_n = \frac{b}{l^2}\left[\cancel{\frac{-l\cos\frac{n\pi}{2}}{\left(\frac{n\pi}{2l}\right)}} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2l}\right)^2} + \cancel{\frac{l\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2l}\right)}} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2l}\right)^2}\right]$$

$$b_n = \frac{b}{\cancel{l^2}}\left[2 \times \frac{4\cancel{l^2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$b_n = \frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \begin{cases} \frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right), & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Sub. $b_n$ in (4)

$$y = \sum_{n=odd}^{\infty} \frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{2l}\right)\cos\left(\frac{n\pi at}{2l}\right)$$

$$\therefore y = \frac{8b}{\pi^2}\sum_{n=odd}^{\infty} \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{2l}\right)\cos\left(\frac{n\pi at}{2l}\right)$$

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TYPE : 2 NON - ZERO VELOCITY

The conditions are

i) $y=0$ when $x=0$

ii) $y=0$ when $x=l$ or $2l$

iii) $y=0$ when $t=0$

iv) $\frac{\partial y}{\partial t} = f(x)$ when $t=0$

WORKING RULE : NON - ZERO VELOCITY

Step: 1 $A=0$

step: 2 $p = \frac{n\pi}{l}$

step: 3 $C=0$

step: 4 $y = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi a t}{l}\right)$

Problem: 6

A tightly stretched string of length '$l$' is initially at rest in its equilibrium position and each of it's given the velocity $v_0 \sin^3\left(\frac{\pi x}{l}\right)$. Find displacement $y(x,t)$.

* Solution:

The one dimensional wave eqn. is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The solution is

$$y = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \longrightarrow (1)$$

The conditions are

i) $y=0$ when $x=0$

ii) $y=0$ when $x=l$

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iii) $y = 0$ when $t = 0$

iv) $\quad$ iv) $\frac{\partial y}{\partial t} = v_0 \sin^3\left(\frac{\pi x}{l}\right)$ when $t = 0$

Step:1 Applying cond. (i) in (1)

$y = 0$ when $x = 0$

$$0 = (A\cos 0 + B\sin 0)(c\cos pat + D\sin pat)$$

$$0 = A(c\cos pat + D\sin pat)$$

If $(c\cos pat + D\sin pat) \neq 0$

$$\therefore A = 0$$

Sub. $A = 0$ in (1)

$$y = B\sin px\,(c\cos pat + D\sin pat) \longrightarrow (2)$$

Step:2 Applying cond. (ii) in (2)

$y = 0$ when $x = l$

$$0 = B\sin pl\,(c\cos pat + D\sin pat)$$

If $B \neq 0$, $(c\cos pat + D\sin pat) \neq 0$

$$\therefore \sin pl = 0$$

$$pl = \sin^{-1}(0) = n\pi$$

$$p = \frac{n\pi}{l}$$

Sub. $p = \frac{n\pi}{l}$ in (2)

$$y = B\sin\left(\frac{n\pi x}{l}\right)\left[c\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right] \longrightarrow (3)$$

Step:3 Applying cond. (iii) in (3)

$y = 0$ when $t = 0$

$$0 = B\sin\left(\frac{n\pi x}{l}\right)\left[c\cos 0 + D\sin 0\right]$$

$$0 = Bc\sin\left(\frac{n\pi x}{l}\right)$$

If $B \neq 0$, $\sin\left(\frac{n\pi x}{l}\right) \neq 0$

$$\therefore c = 0$$

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Put $c=0$ in (3)

$$y = B\sin\left(\frac{n\pi x}{l}\right)\cdot D\sin\left(\frac{n\pi at}{l}\right)$$

$$y = BD\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi at}{l}\right)$$

Put $BD = C_n$

$$y = C_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi at}{l}\right)$$

The most general eqn. is

$$y = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi at}{l}\right) \longrightarrow (4)$$

Step: 4

Partial Diff. (4) w.r.t. to 't'

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi at}{l}\right)\cdot\left(\frac{n\pi a}{l}\right)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} C_n\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)$$

Applying cond. (iv) in (4)

$$\frac{\partial y}{\partial t} = f(x) \text{ when } t=0$$

$$f(x) = \sum_{n=1}^{\infty} C_n\left(\frac{n\pi a}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\cos 0$$

Put $b_n = C_n \cdot \frac{n\pi a}{l}$

$$V_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$$

$$\frac{V_0}{4}\left[3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right] = b_1\sin\left(\frac{\pi x}{l}\right) + b_2\sin\left(\frac{2\pi x}{l}\right)$$

$$+ b_3\sin\left(\frac{3\pi x}{l}\right) + \cdots$$

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Equating the coefficients of $\sin\left(\frac{\pi x}{l}\right), \sin\left(\frac{2\pi x}{l}\right), \sin\left(\frac{3\pi x}{l}\right) \dots$

$$b_1 = \frac{3V_0}{4}$$

$$b_3 = \frac{-V_0}{4}$$

$$b_2 = b_4 = b_5 = \dots = 0$$

From $b_n = C_n\left(\frac{n\pi a}{l}\right)$

Put $n=1$

$$b_1 = C_1\left(\frac{\pi a}{l}\right)$$

$$C_1 = b_1\left(\frac{l}{\pi a}\right)$$

$$C_1 = \frac{3V_0 l}{4\pi a}$$

$n=3$

$$b_3 = C_3\left(\frac{3\pi a}{l}\right)$$

$$C_3 = b_3\left(\frac{l}{3\pi a}\right)$$

$$C_3 = \frac{-V_0 l}{12\pi a}$$

From (4)

$$y = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi a t}{l}\right)$$

$$y = \frac{3V_0 l}{4\pi a}\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi a t}{l}\right) - \frac{V_0 l}{12\pi a}\sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3\pi a t}{l}\right).$$

Problem:6

A tightly stretched string of length '$l$' is initially at rest in its equilibrium position and each of it's given the velocity $V_0 \sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)$. Find the displacement $y(x,t)$.

*Solution:

The one dimensional wave eqn. is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

The solution is

$$y = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \longrightarrow (1)$$

The conditions are

i) $y=0$ when $x=0$

ii) $y=0$ when $x=l$

iii) $y=0$ when $t=0$

iv) $\frac{\partial y}{\partial t} = V_0 \sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)$ when $t=0$.

Step: 1 Applying cond. (i) in (1)

$y=0$ when $x=0$

$$0 = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$0 = A(C\cos pat + D\sin pat)$$

If $(C\cos pat + D\sin pat) \neq 0$

$\therefore A=0$

Sub. $A=0$ in (1)

$$y = B\sin px(C\cos pat + D\sin pat) \longrightarrow (2)$$

Step: 2 Applying cond. (ii) in (2)

$$0 = B\sin pl\,(C\cos pat + D\sin pat)$$

If $B \neq 0$, $(C\cos pat + D\sin pat) \neq 0$

$\therefore \sin pl = 0$

$pl = \sin^{-1}(0) = n\pi$

$p = \frac{n\pi}{l}$

Sub. $p = \frac{n\pi}{l}$ in (2)

$$y = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right] \longrightarrow (3)$$

Step: 3 Applying cond. (iii) in (3)

$y = 0$ when $t = 0$

$$0 = B \sin\left(\frac{n\pi x}{l}\right)\left[c\cos 0 + D \sin 0\right]$$

$$0 = BC \sin\left(\frac{n\pi x}{l}\right)$$

If $B \neq 0$, $\sin\left(\frac{n\pi x}{l}\right) \neq 0$

$\therefore c = 0$

Sub. $c = 0$ in (3)

$$y = B \sin\left(\frac{n\pi x}{l}\right)\left[0 + D \sin\left(\frac{n\pi a t}{l}\right)\right]$$

$$y = BD \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi a t}{l}\right)$$

Put $BD = C_n$

$$y = C_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi a t}{l}\right)$$

The most general eqn. is

$$y = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi a t}{l}\right) \longrightarrow (4)$$

step: 4

Partial Diff. (4) w.r.to 't'

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} C_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right)$$

Applying cond. (iv) in above eqn.

$$\frac{\partial y}{\partial t} = f(x) = V_0 \sin\left(\frac{3\pi x}{l}\right) \cos\left(\frac{\pi x}{l}\right) \text{ when } t = 0$$

$$f(x) = \sum_{n=1}^{\infty} C_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cos 0$$

$$V_0 \sin\left(\frac{3\pi x}{l}\right) \cos\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} C_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

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Put $b_n = C_n \cdot \left(\frac{n\pi a}{l}\right)$

$$V_0 \sin\left(\frac{3\pi x}{l}\right) \cos\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$V_0 \sin\left(\frac{3\pi x}{l}\right) \cos\left(\frac{\pi x}{l}\right) = b_1 \sin\left(\frac{\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right) + b_3 \sin\left(\frac{3\pi x}{l}\right) + \cdots$$

Equating coefficient of $\sin\left(\frac{3\pi x}{l}\right)$

$$b_3 = V_0 \cos\left(\frac{\pi x}{l}\right)$$

Equating coeff. of $\sin\left(\frac{\pi x}{l}\right), \sin\left(\frac{2\pi x}{l}\right) \cdots$

$$b_1 = b_2 = b_4 = b_5 = \cdots = 0$$

From $b_n = C_n \left(\frac{n\pi a}{l}\right)$

Put $n = 3$

$$b_3 = C_3 \left(\frac{3\pi a}{l}\right)$$

$$C_3 = b_3 \left(\frac{l}{3\pi a}\right)$$

$$C_3 = V_0 \cos\left(\frac{\pi x}{l}\right) \left(\frac{l}{3\pi a}\right)$$

From (4)

$$y = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi a t}{l}\right)$$

$$y = C_1 \sin\left(\frac{\pi x}{l}\right) \sin\left(\frac{\pi a t}{l}\right) + C_2 \sin\left(\frac{2\pi x}{l}\right) \sin\left(\frac{2\pi a t}{l}\right) + C_3 \sin\left(\frac{3\pi x}{l}\right) \sin\left(\frac{3\pi a t}{l}\right) + \cdots$$

$$y = \frac{V_0 l}{3\pi a} \cos\left(\frac{\pi x}{l}\right) \sin\left(\frac{3\pi x}{l}\right) \sin\left(\frac{3\pi a t}{l}\right).$$

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Problem: 7

A tightly stretched strings has its end fixed $x=0$ and $x=l$ at time $t=0$ initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l-x)$, show that

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi x}{l} \sin\frac{(2n-1)\pi a t}{l}$$

* solution:

The one dimensional wave eqn. is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The solution is

$$y = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \longrightarrow (1)$$

The conditions are

(i) $y=0$ when $x=0$

ii) $y=0$ when $x=l$

iii) $y=0$ when $t=0$

iv) $\frac{\partial y}{\partial t} = f(x) = \lambda x(l-x)$ when $t=0$

Step: 1 Applying cond. (i) in (1)

$y=0$ when $x=0$

$$0 = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$0 = A(C\cos pat + D\sin pat)$$

If $(C\cos pat + D\sin pat) \neq 0$

$\therefore A = 0$

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Sub. $A=0$ in (1)

$$y = B\sin px\,(C\cos pat + D\sin pat) \longrightarrow (2)$$

Step: 2 Applying cond (ii) in (2)

$y=0$ when $x=l$

$$0 = B\sin pl\,(C\cos pat + D\sin pat)$$

If $B \neq 0$, $(C\cos pat + D\sin pat) \neq 0$

$\therefore \sin pl = 0$

$pl = \sin^{-1}(0) = n\pi$

$p = \frac{n\pi}{l}$

Sub. $p = \frac{n\pi}{l}$ in (2)

$$y = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right] \longrightarrow (3)$$

Step: 3 Applying cond. (iii) in (3)

$y=0$ when $t=0$

$$0 = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos 0 + D\sin 0\right]$$

$$0 = BC\sin\left(\frac{n\pi x}{l}\right)$$

If $B \neq 0$, $\sin\left(\frac{n\pi x}{l}\right) \neq 0$

$\therefore C = 0$

Sub. $C=0$ in (3)

$$y = B\sin\left(\frac{n\pi x}{l}\right)\left[0 + D\sin\left(\frac{n\pi at}{l}\right)\right]$$

$$y = BD\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi at}{l}\right)$$

Put $BD = C_n$

$$y = C_n\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi at}{l}\right)$$

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The most general eqn. is

Step: 4 $y = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi a t}{l}\right) \longrightarrow (4)$

Partial Diff. w.r.t. to 't'

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} C_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right)$$

Applying cond. (iv) in above eqn.

$$\frac{\partial y}{\partial t} = f(x) = \lambda x(l-x) \quad \text{when } t=0$$

$$f(x) = \sum_{n=1}^{\infty} C_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cos 0.$$

$$f(x) = \sum_{n=1}^{\infty} C_n \left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

Put $b_n = C_n \left(\frac{n\pi a}{l}\right)$

$$\lambda x(l-x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Here R.H.S represents H.R.S.S

To find $b_n$:

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{2}{l}\int_0^l \lambda x(l-x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{\lambda 2}{l}\int_0^l (lx - x^2) \sin\left(\frac{n\pi x}{l}\right) dx$$

Using Bernoulli's theorem,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

$u = lx - x^2 \qquad v = \sin\left(\frac{n\pi x}{l}\right)$

$u' = l - 2x \qquad v_1 = \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} \qquad v_3 = \frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}$

$u'' = -2$

$u''' = 0 \qquad v_2 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}$

$$b_n = \frac{2\lambda}{l}\left[-(lx-x^2)\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} + (l-2x)\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} - 2\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l$$

$$b_n = \frac{2\lambda}{l}\left[-\frac{2\cos n\pi}{\left(\frac{n\pi}{l}\right)^3} + \frac{2\cos 0}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$b_n = \frac{2\lambda}{\cancel{l}} \times \frac{l^3}{n^3\pi^3}\left[-2(-1)^n + 2\right]$$

$$b_n = \frac{2\lambda l^2 \times 2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} \frac{8\lambda l^2}{n^3\pi^3}, & \text{if } n = \text{odd} \\ 0, & \text{if } n = \text{even} \end{cases}$$

From

$$C_n = b_n\left(\frac{l}{n\pi a}\right)$$

$$C_n = \begin{cases} \frac{8\lambda l^2}{n^3\pi^3}\left(\frac{l}{n\pi a}\right), & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Sub. $C_n$ in (4)

$$y = \sum_{n=odd}^{\infty} \frac{8\lambda l^3}{n^4\pi^4 a}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi a t}{l}\right)$$

$$y = \frac{8\lambda l^3}{\pi^4 a}\sum_{n=odd}^{\infty}\frac{1}{(n)^4}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi a t}{l}\right)$$

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Replace $n$ by $(2n-1)$

$$\therefore \quad y = \frac{8\lambda l^3}{\pi^4 a}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\left(\frac{(2n-1)\pi x}{l}\right) \sin\left(\frac{(2n-1)\pi a t}{l}\right).$$

TWO DIMENSIONAL HEAT FLOW EQN:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

* THREE POSSIBLE SOLUTIONS:

i) $u = (A\cos px + B\sin px)(Ce^{+py} + De^{-py}) \rightarrow$ x-direction (or) Horizontal

ii) $u = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)$

y-direction (or) vertical

iii) $u = (Ax+B)(Cy+D)$

Working rule to form boundary conditions:

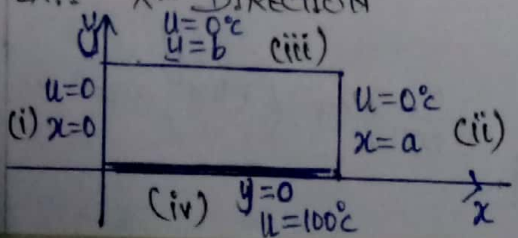
1.) 3 edges are kept at zero temperature and 1 edge is kept at non-zero temperature.

2.) The 4th condition is the edge of the non-zero temperature.

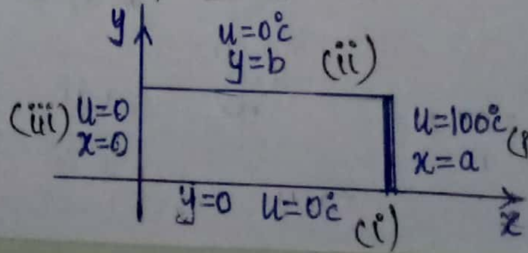
3.) The 3rd condition is opposite to 4th one.

4.) 1st condition always along axis and another one is second condition.

EX:1 X- DIRECTION




EX-2: Y- DIRECTION

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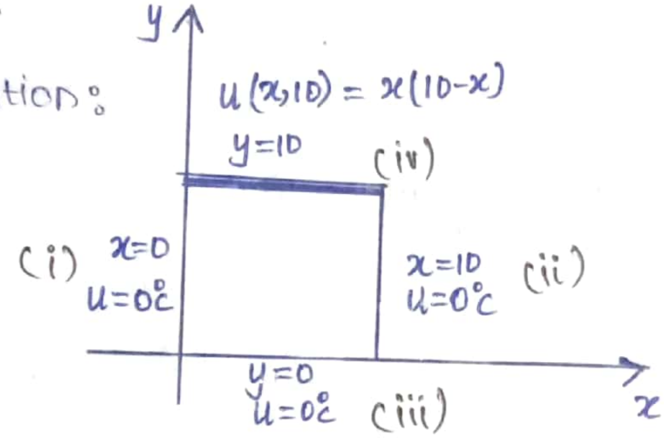
Type : 1 FINITE PLATE

Problem : 8

A square plate is bounded by the lines $x=0$, $y=0$, $x=10$ and $y=10$. It's faces are insulated. The temp along the upper horizontal edge is given by $u(x,10) = x(10-x)$, $0<x<10$ while other edges are kept at 0°c. Find the steady state temp. distribution in the plate.

* Solution :




The two dimensional heat eqn. is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The general solution is

$$u = (A\cos px + B\sin px)(ce^{py} + De^{-py}) \longrightarrow (1)$$

x- direction

The conditions are

i) $u=0$ when $x=0$

ii) $u=0$ when $x=10$

iii) $u=0$ when $y=0$

iv) $u = x(10-x)$ when $y=10$

Step:1 Applying cond. (i) in (I)

$u=0$ when $x=0$

$$0 = (A\cos 0 + B\sin 0)(Ce^{py} + De^{-py})$$

$$0 = A(Ce^{py} + De^{-py})$$

If $Ce^{py} + De^{-py} \neq 0$

$\therefore A=0$

Sub. $A=0$ in (I)

$$u = B\sin px(Ce^{py} + De^{-py}) \longrightarrow (2)$$

Step:2 Applying cond. (ii) in (2)

$u=0$ when $x=10$

$$0 = B\sin 10p(Ce^{py} + De^{-py})$$

If $B\neq 0$, $(Ce^{py} + De^{-py}) \neq 0$

$\therefore \sin 10p = 0$

$10p = \sin^{-1}(0) = n\pi$

$p = \frac{n\pi}{10}$

Sub. $p = \frac{n\pi}{10}$ in (2)

$$u = B\sin\left(\frac{n\pi x}{10}\right)\left[Ce^{\frac{n\pi y}{10}} + De^{-\frac{n\pi y}{10}}\right] \longrightarrow (3)$$

Step:3 Applying cond. (iii) in (3)

$u=0$ when $y=0$

$$0 = B\sin\left(\frac{n\pi x}{10}\right)\left[Ce^{0} + De^{0}\right]$$

$$0 = B\sin\left(\frac{n\pi x}{10}\right)\left[C+D\right]$$

If $B\neq 0$, $\sin\left(\frac{n\pi x}{10}\right) \neq 0$

$\therefore C+D=0 \Rightarrow D=-C$

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Sub. $D = -C$ in (3)

$$u = B \sin\left(\frac{n\pi x}{10}\right)\left[Ce^{\frac{n\pi y}{10}} - Ce^{-\frac{n\pi y}{10}}\right]$$

$$u = BC \sin\left(\frac{n\pi x}{10}\right)\left[e^{\frac{n\pi y}{10}} - e^{-\frac{n\pi y}{10}}\right]$$

Put $BC = C_n$

$$u = C_n \sin\left(\frac{n\pi x}{10}\right)\left[e^{\frac{n\pi y}{10}} - e^{-\frac{n\pi y}{10}}\right]$$

The most general solution is

$$u = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{10}\right)\left[e^{\frac{n\pi y}{10}} - e^{-\frac{n\pi y}{10}}\right] \rightarrow (4)$$

Step: 4 Applying cond. (iv) in (4)

$u = x(10-x)$ when $y = 10$

$$x(10-x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{10}\right)\left[e^{n\pi} - e^{-n\pi}\right]$$

Put $b_n = C_n(e^{n\pi} - e^{-n\pi})$

$$x(10-x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)$$

$\therefore$ R.H.S represents H.R.S.S.

To find $b_n$:

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Here $l = 10$

$$b_n = \frac{2}{10}\int_0^{10} x(10-x) \sin\left(\frac{n\pi x}{10}\right) dx$$

$$b_n = \frac{1}{5}\int_0^{10} (10x - x^2) \sin\left(\frac{n\pi x}{10}\right) dx$$

By using Bernoulli's theorem,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

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$u = 10x - x^2$ $\quad v = \sin\left(\frac{n\pi x}{10}\right)$

$u' = 10 - 2x$ $\quad v_1 = \frac{-\cos\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)}$

$u'' = -2$ $\quad v_2 = \frac{-\sin\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}$

$u''' = 0$ $\quad v_3 = \frac{\cos\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^3}$

$$b_n = \frac{1}{5}\left[-(10x - x^2)\frac{\cos\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)} + (10-2x)\frac{\sin\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2} - \frac{2\cos\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^3}\right]_0^{10}$$

$$b_n = \frac{1}{5}\left[\frac{-2\cos n\pi}{\left(\frac{n\pi}{10}\right)^3} + \frac{2\cos 0}{(n\pi/10)^3}\right]$$

$$= \frac{2}{5} \times \frac{(10)^3}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$= \frac{2 \times \cancel{1000}^{200}}{\cancel{5} n^3\pi^3}\left(1 - (-1)^n\right)$$

$$b_n = \frac{400}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$b_n = \begin{cases} \frac{800}{n^3\pi^3}, & \text{if } n = \text{odd} \\ 0, & \text{if } n = \text{even} \end{cases}$$

From $\quad b_n = C_n\left(e^{n\pi} - e^{-n\pi}\right)$

$$C_n = \frac{b_n}{\left(e^{n\pi} - e^{-n\pi}\right)}$$

$$C_n = \begin{cases} \frac{800}{n^3\pi^3\left(e^{n\pi} - e^{-n\pi}\right)}, & \text{if } n = \text{odd} \\ 0, & \text{if } n = \text{even} \end{cases}$$

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sub. $C_n$ in (4)

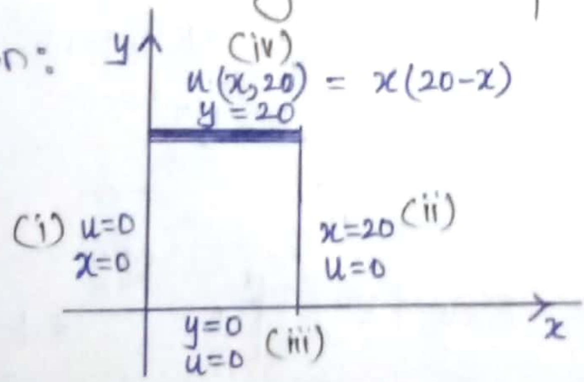
$$u = \sum_{n=odd}^{\infty} \frac{800}{n^3\pi^3(e^{n\pi}-e^{-n\pi})} \sin\left(\frac{n\pi x}{10}\right) \cdot \left[ e^{\frac{n\pi y}{10}} - e^{-\frac{n\pi y}{10}} \right]$$

$$u = \frac{800}{\pi^3} \sum_{n=odd}^{\infty} \left[ \frac{e^{\frac{n\pi y}{10}} - e^{-\frac{n\pi y}{10}}}{n^3 (e^{n\pi} - e^{-n\pi})} \right] \cdot \sin\left(\frac{n\pi x}{10}\right).$$

Problem: 9

A square plate is bounded by the lines $x=0$, $y=0$, $x=20$ and $y=20$. Its faces are insulated. The temp. along the upper horizontal edge is given by $u(x,20) = x(20-x)$, $0<x<20$ while other edges are kept at 0°c. Find steady state temp. in the plate.

* Solution:




The two dimensional heat eqn. is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The general solution is

$$u = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \longrightarrow (1)$$

The conditions are

i) $u=0$ when $x=0$

ii) $u=0$ when $x=20$

iii) $u=0$ when $y=0$

iv) $u = x(20-x)$ when $y=20$

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Step: 1 Applying cond (i) in (1)

$u = 0$ when $x = 0$

$0 = (A\cos 0 + B\sin 0)(ce^{py} + De^{-py})$

$0 = A(ce^{py} + De^{-py})$

If $ce^{py} + De^{-py} \neq 0$

$\therefore A = 0$

Sub. $A = 0$ in (1)

$u = B\sin px(ce^{py} + De^{-py}) \longrightarrow (2)$

Step: 2 Applying cond. (ii) in (2)

$u = 0$ when $x = 20$

$0 = B\sin 20p\,(ce^{py} + De^{-py})$

If $B \neq 0$, $ce^{py} + De^{-py} \neq 0$

$\sin 20p = 0$

$20p = \sin^{-1}(0) = n\pi$

$p = \frac{n\pi}{20}$

Sub. $p = \frac{n\pi}{20}$ in (2)

$u = B\sin\left(\frac{n\pi x}{20}\right)\left[ce^{\frac{n\pi y}{20}} + De^{-\frac{n\pi y}{20}}\right] \longrightarrow (3)$

Step: 3 Applying cond. (iii) in (3)

$u = 0$ when $y = 0$

$0 = B\sin\left(\frac{n\pi x}{20}\right)\left[ce^{0} + De^{0}\right]$

$0 = B\sin\left(\frac{n\pi x}{20}\right)[c + D]$

If $B \neq 0$, $\sin\left(\frac{n\pi x}{20}\right) \neq 0$

$\therefore c + D = 0$

$D = -c$

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Sub. $D = -c$ in (3)

$$u = B\sin\left(\frac{n\pi x}{20}\right)\left[ce^{\frac{n\pi y}{20}} - ce^{-\frac{n\pi y}{20}}\right]$$

$$u = Bc\sin\left(\frac{n\pi x}{20}\right)\left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right]$$

Put $Bc = C_n$

$$u = C_n\sin\left(\frac{n\pi x}{20}\right)\left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right]$$

The most general solution is

$$u = \sum_{n=1}^{\infty} C_n\sin\left(\frac{n\pi x}{20}\right)\left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right] \longrightarrow (4)$$

Step: 4 Applying condn. (iv) in (4)

$u = x(20-x)$ when $y = 20$

$$x(20-x) = \sum_{n=1}^{\infty} C_n\sin\left(\frac{n\pi x}{20}\right)\left[e^{n\pi} - e^{-n\pi}\right]$$

Put $b_n = C_n\left[e^{n\pi} - e^{-n\pi}\right]$

$$f(x) = x(20-x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{20}\right)$$

Here R.H.S represents H.R.S.S.

To find $b_n$:

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Here $l = 20$

$$b_n = \frac{2}{20}\int_0^l x(20-x)\sin\left(\frac{n\pi x}{20}\right)dx$$

$$b_n = \frac{1}{10}\int_0^{20}(20x - x^2)\sin\left(\frac{n\pi x}{20}\right)dx$$

By using Bernoulli's theorem,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

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$u = 20x - x^2$ $\quad v = \sin\left(\frac{n\pi x}{20}\right)$ $\quad v_3 = \frac{\cos\left(\frac{n\pi x}{20}\right)}{\left(\frac{n\pi}{20}\right)^3}$

$u' = 20 - 2x$ $\quad v_1 = \frac{-\cos\left(\frac{n\pi x}{20}\right)}{\left(n\pi/20\right)}$

$u'' = -2$

$u''' = 0$ $\quad v_2 = \frac{-\sin\left(\frac{n\pi x}{20}\right)}{\left(n\pi/20\right)^2}$

$$b_n = \frac{1}{10}\left[-(20x-x^2)\frac{\cos\left(\frac{n\pi x}{20}\right)}{\left(\frac{n\pi}{20}\right)} + (20-2x)\frac{\sin\left(\frac{n\pi x}{20}\right)}{\left(\frac{n\pi}{20}\right)^2} - \frac{2\cos\left(\frac{n\pi x}{20}\right)}{\left(\frac{n\pi}{20}\right)^3}\right]_0^{20}$$

$$b_n = \frac{1}{10}\left[-\frac{2\cos n\pi}{\left(\frac{n\pi}{20}\right)^3} + \frac{2\cos 0}{\left(\frac{n\pi}{20}\right)^3}\right]$$

$$= \frac{1}{10} \times \frac{(20)^3}{n^3\pi^3} \times 2\left[1-(-1)^n\right]$$

$$= \frac{8000 \times 2}{10\, n^3\pi^3}\left(1-(-1)^n\right]$$

$$b_n = \frac{1600}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$b_n = \begin{cases} \frac{3200}{n^3\pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

From $b_n = C_n\left(e^{n\pi} - e^{-n\pi}\right)$

$$C_n = \frac{b_n}{\left(e^{n\pi} - e^{-n\pi}\right)}$$

$$C_n = \frac{3200}{n^3\pi^3\left(e^{n\pi} - e^{-n\pi}\right)}$$

Sub. $C_n$ in (4)

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$$u = \sum_{n=odd}^{\infty} \frac{3200}{n^3\pi^3(e^{n\pi} - e^{-n\pi})} \sin\left(\frac{n\pi x}{20}\right)\left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right]$$

$$u = \frac{3200}{\pi^3}\sum_{n=odd}^{\infty} \frac{\left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right]}{n^3\left[e^{n\pi} - e^{-n\pi}\right]} \sin\left(\frac{n\pi x}{10}\right)$$

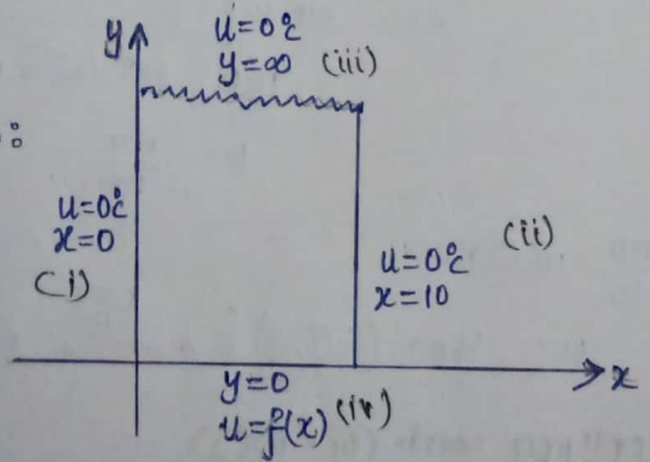
Type: 2 Infinite plate
Problem: 10

An infinitely long rectangular plate with insulated surface is 10 cm wide. The two long edges and one short edge are kept at zero temperature while the other short edge $y=0$ is kept at temp. given by

$$u = \begin{cases} 20x & , 0 \le x \le 5 \\ 20(10-x) & , 5 \le x \le 10 \end{cases}$$

Find steady state temp. in the plate.

* solution:




The two Dimensional heat flow eqn. is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The solution is

$$u = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \longrightarrow ①$$

The conditions are

i) $u=0$ when $x=0$
ii) $u=0$ when $x=10$
iii) $u=0$ when $y=\infty$
iv) $u=f(x)$ when $y=0$

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Step: 1 Applying cond. ① in ①

$u = 0$ when $x = 0$

$0 = (A + B\sin 0)(ce^{py} + De^{-py})$

$0 = A(ce^{py} + De^{-py})$

If $ce^{py} + De^{-py} \neq 0 \quad \therefore A = 0$

Sub. $A = 0$ in ①

$u = B\sin px\,(ce^{py} + De^{-py}) \longrightarrow ②$

Step: 2 Applying cond. (ii) in ②

$u = 0$ when $x = 10$

$0 = B\sin 10p\,(ce^{py} + De^{-py})$

If $B \neq 0$, $ce^{py} + De^{-py} \neq 0$

$\therefore \sin 10p = 0$

$10p = \sin^{-1}(0) = n\pi$

$p = \frac{n\pi}{10}$

Sub. $p = \frac{n\pi}{10}$ in ②

$u = B\sin\left(\frac{n\pi x}{10}\right)\left[ce^{\frac{n\pi y}{10}} + De^{-\frac{n\pi y}{10}}\right] \longrightarrow ③$

Step: 3 Applying cond. (iii) in ③

$u = 0$ when $y = \infty$

$0 = B\sin\left(\frac{n\pi x}{10}\right)\left[ce^{\infty} + De^{-\infty}\right] \quad \because e^{-\infty} = 0$

$0 = B\sin\left(\frac{n\pi x}{10}\right)ce^{\infty}$

If $B \neq 0$, $e^{\infty} \neq 0$, $\sin\left(\frac{n\pi x}{10}\right) \neq 0$

$\therefore c = 0$

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Sub. $c=0$ in ③

$$u = B \sin\left(\frac{n\pi x}{10}\right) \cdot D e^{-\frac{n\pi y}{10}}$$

$$u = BD\, e^{-\frac{n\pi y}{10}} \sin\left(\frac{n\pi x}{10}\right)$$

Sub. $BD = b_n$

$$u = b_n \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n\pi y}{10}}$$

The most general solution is

$$u = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n\pi y}{10}} \longrightarrow ④$$

Step: 4 Applying cond. (iv) in ④

$$u = f(x) \text{ when } y=0$$

$$u = f(x) = \begin{cases} 20x & , \quad 0 \le x \le 5 \\ 20(10-x) & , \quad 5 \le x \le 10 \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) e^{0}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)$$

R.H.S represents H.R.S.S

To find $b_n$:

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Here $l = 10$

$$b_n = \frac{2}{10} \int_0^{10} f(x) \sin\left(\frac{n\pi x}{10}\right) dx$$

$$b_n = \frac{1}{5}\left\{ \int_0^5 20x \sin\left(\frac{n\pi x}{10}\right) dx + \int_5^{10} 20(10-x) \sin\left(\frac{n\pi x}{10}\right) dx \right\}$$

By using Bernoulli's theorem,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

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$u = x \qquad u = 10 - x \qquad v = \sin\left(\frac{n\pi x}{10}\right)$

$u' = 1 \qquad u' = -1 \qquad v_1 = \frac{-\cos\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)}$

$u'' = 0 \qquad u'' = 0$

$v_2 = \frac{-\sin\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}$

$$b_n = \frac{20}{5}\left\{\left[\frac{-x\cos\left(\frac{n\pi x}{10}\right)}{\frac{n\pi}{10}} + \frac{\sin\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]_0^5 + \left[\frac{-(10-x)\cos\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)} - \frac{\sin\left(\frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]_5^{10}\right\}$$

$$b_n = 4\left\{\cancel{\frac{-5\cos\left(\frac{n\pi 5}{10}\right)}{\left(\frac{n\pi}{10}\right)}} + \frac{\sin\left(\frac{n\pi 5}{10}\right)}{\left(\frac{n\pi}{10}\right)^2} + \cancel{\frac{5\cos\left(\frac{n\pi 5}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}} + \frac{\sin\left(\frac{n\pi 5}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right\}$$

$$b_n = 4 \times \frac{(10)^2}{n^2\pi^2} \times 2\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{800}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \begin{cases} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n = \text{even} \end{cases}$$

Sub $b_n$ in (4)

$$u = \sum_{n=odd}^{\infty} \frac{800}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{10}\right)e^{-\frac{n\pi y}{10}}$$

$$u = \frac{800}{\pi^2}\sum_{n=odd}^{\infty}\frac{e^{-\frac{n\pi y}{10}}}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{10}\right).$$

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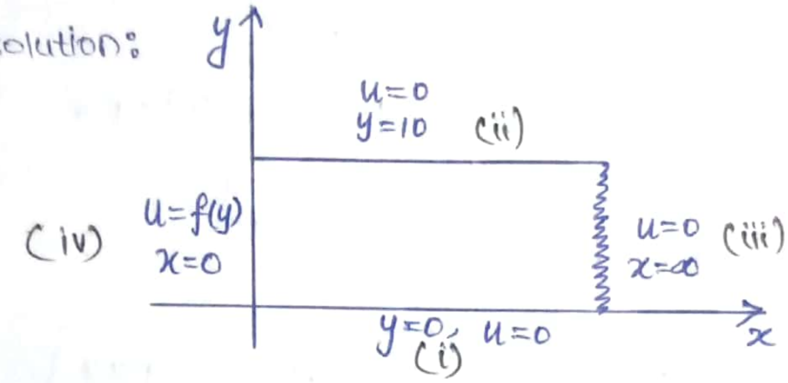
Problem : 11

An infinitely long rectangular plate with insulated surface 10 cm wide. The two long edges and one short edge are kept at zero temp. while the other short edge $x=0$ is kept at temp. given by $u = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10-y), & 5 \le y \le 10 \end{cases}$

Find the steady state temp. in the plate.

* solution:




The two dimensional heat flow eqn. is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The solution is

$$u = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \longrightarrow (1)$$

The conditions are

i) $u=0$ when $y=0$

ii) $u=0$ when $y=10$

iii) $u=0$ when $x=\infty$

iv) $u=f(y)$ when $x=0$

Step: 1 Applying cond. (i) in (1)

$u=0$ when $y=0$

$$0 = (Ae^{px} + Be^{-px})(C\cos 0 + D\sin 0)$$

$$0 = C(Ae^{px} + Be^{-px})$$

If $Ae^{px} + Be^{-px} \neq 0$ $\therefore C=0$

Sub. $c=0$ in (1)

$$u = (Ae^{px} + Be^{-px}) D \sin py \longrightarrow (2)$$

Step: 2 Applying cond. (ii) in (2)

$u=0$ when $y=10$

$$0 = D \sin 10p (Ae^{px} + Be^{-px})$$

If $D \neq 0$, $(Ae^{px} + Be^{-px}) \neq 0$

$\therefore \sin 10p = 0$

$10p = \sin^{-1}(0) = n\pi$

$p = \frac{n\pi}{10}$

Sub. $p = \frac{n\pi}{10}$ in (2)

$$u = \left(Ae^{\frac{n\pi x}{10}} + Be^{-\frac{n\pi x}{10}}\right) \cdot D \sin\left(\frac{n\pi y}{10}\right) \longrightarrow (3)$$

Step: 3 Applying cond. (iii) in (3)

$u=0$ when $x=\infty$

$$0 = (Ae^{\infty} + Be^{-\infty}) D \sin\left(\frac{n\pi y}{10}\right) \qquad e^{-\infty}=0$$

If $D \neq 0$ $\qquad 0 = ADe^{\infty} \sin\left(\frac{n\pi y}{10}\right)$

$\sin\left(\frac{n\pi y}{10}\right) \neq 0$ $\qquad \therefore A=0$

$e^{\infty} \neq 0$

Sub. $A=0$ in (3)

$$u = Be^{-\frac{n\pi x}{10}} \cdot D \sin\left(\frac{n\pi y}{10}\right)$$

$$u = BDe^{-\frac{n\pi x}{10}} \sin\left(\frac{n\pi y}{10}\right)$$

Put $BD = b_n$

$$u = b_n e^{-\frac{n\pi x}{10}} \sin\left(\frac{n\pi y}{10}\right)$$

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The most general solution is

$$u = \sum_{n=1}^{\infty} b_n e^{-\frac{n\pi x}{10}} \sin\left(\frac{n\pi y}{10}\right) \longrightarrow (4)$$

Step: 4 Applying condn. (iv) in (4)

$u = f(y)$ when $x = 0$

$$f(y) = \begin{cases} 20y & , 0 \le y \le 5 \\ 20(10-y) & , 5 \le y \le 10 \end{cases}$$

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{10}\right) e^{0}$$

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{10}\right)$$

Here R.H.S represents H.R.S.S

To find $b_n$:

$$b_n = \frac{2}{l} \int_0^l f(y) \sin\left(\frac{n\pi y}{l}\right) dy$$

Here $l = 10$

$$b_n = \frac{2}{10} \int_0^{10} f(y) \sin\left(\frac{n\pi y}{10}\right) dy$$

$$b_n = \frac{1}{5} \left\{ \int_0^5 20y \sin\left(\frac{n\pi y}{10}\right) dy + \int_5^{10} 20(10-y) \sin\left(\frac{n\pi y}{10}\right) dy \right\}$$

By using Bernoulli's theorem,

$\therefore \int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

$u = y$ $\quad v = \sin\left(\frac{n\pi y}{10}\right)$ $\quad u = 10 - y$

$u' = 1$ $\quad v_1 = \dfrac{-\cos\left(\frac{n\pi y}{10}\right)}{\left(\frac{n\pi}{10}\right)}$ $\quad u' = -1$

$u'' = 0$ $\quad u'' = 0$

$$v_2 = -\frac{\sin\left(\frac{n\pi y}{10}\right)}{(n\pi/10)^2}$$

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$$b_n = \frac{20}{5}\left\{\left[\frac{-y\cos\left(\frac{n\pi y}{10}\right)}{\left(\frac{n\pi}{10}\right)} + \frac{\sin\left(\frac{n\pi y}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]_0^5 + \left[\frac{-(10-y)\cos\left(\frac{n\pi y}{10}\right)}{\left(\frac{n\pi}{10}\right)} - \frac{\sin\left(\frac{n\pi y}{10}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]_5^{10}\right\}$$

$$b_n = 4\left[\frac{-5\cos\left(\frac{n\pi 5}{10}\right)}{\left(\frac{n\pi}{10}\right)} + \frac{\sin\left(\frac{n\pi 5}{10_2}\right)}{\left(\frac{n\pi}{10}\right)^2} + \frac{5\cos\left(\frac{n\pi 5}{10}\right)}{\left(\frac{n\pi}{10}\right)} + \frac{\sin\left(\frac{n\pi 5}{10_2}\right)}{\left(\frac{n\pi}{10}\right)^2}\right]$$

$$b_n = \frac{4\times 100}{n^2\pi^2}\cdot 2\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \begin{cases} \frac{800}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) & \text{when } n=\text{odd} \\ 0 & \text{when } n=\text{even} \end{cases}$$

Sub $b_n$ in (4)

$$u = \sum_{n=1}^{\infty} b_n e^{-\frac{n\pi x}{10}} \cdot \sin\left(\frac{n\pi y}{10}\right)$$

$$u = \sum_{n=\text{odd}}^{\infty} \frac{800}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) e^{-\frac{n\pi x}{10}} \sin\left(\frac{n\pi y}{10}\right)$$

$$\therefore\ u = \frac{800}{\pi^2}\sum_{n=\text{odd}}^{\infty} \frac{e^{-\frac{n\pi x}{10}}}{n^2}\sin\left(\frac{n\pi}{2}\right)\cdot\sin\left(\frac{n\pi y}{10}\right.$$

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ONE DIMENSIONAL HEAT EQUATION:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $c^2 \rightarrow$ $c^2 = \dfrac{\text{Thermal Conductivity}}{(\text{Density})(\text{specific heat})}$

* THREE POSSIBLE SOLUTION

i) $u = (Ae^{px} + Be^{-px})\, ce^{\alpha^2 p^2 t}$

ii) $u = (A\cos px + B\sin px)\, ce^{-\alpha^2 p^2 t}$

iii) $u = (Ax + B)c$

The conditions are

i) $u = 0$ when $x = 0$

ii) $u = 0$ when $x = l$

iii) $u = f(x)$ when $t = 0$

Type : 1 (Steady state Condition)

Problem : 13

A rod 30 cm long has its end A and B kept at 20°c and 80°c respectively until steady state condition prevails. The temp. at each end is suddenly reduced to 0°c and kept so. Find resulting temp. function $u(x,t)$, taking $x=0$ at A.

* solution:

One Dimensional heat eqn.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

when steady state prevails

$$\frac{\partial u}{\partial t} = 0 \qquad \Rightarrow 0 = c^2 \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$$

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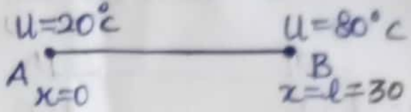
To find temp.:

$$u(x) = ax+b \longrightarrow (A)$$




When $x=0$ $u=20$

$$20 = a(0)+b$$

$$b=20$$

When $x=30$, $u=80$

$$80 = 30a+b$$

$$80 = 30a+20$$

$$30a = 60$$

$$a=2$$

$\therefore$ (A) becomes $u = 2x+20$

The conditions are

i) $u=0$ when $x=0$

ii) $u=0$ when $x=l=30$

iii) $u=f(x)=2x+20$, when $t=0$

The general solution is

$$u = (A\cos px + B\sin px)\, ce^{-\alpha^2p^2t} \longrightarrow (1)$$

Step:1 Applying cond. (i) in (1)

$u=0$ when $x=0$

$$0 = (A\cos 0 + B\sin 0)\, ce^{-\alpha^2p^2t}$$

$$0 = Ace^{-\alpha^2p^2t}$$

If $ce^{-\alpha^2p^2t} \neq 0$, $\therefore A=0$

Sub. $A=0$ in (1)

$$u = BC\sin px.e^{-\alpha^2p^2t} \longrightarrow (2)$$

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**Step: 2** Applying condition (ii) in (2)

$u = 0$ & $x = 30$

$$0 = BC \sin 30p \, e^{-\alpha^2 p^2 t}$$

Here $B \neq 0$, $C \neq 0$ & $e^{-\alpha^2 p^2 t} \neq 0$

$$\therefore \sin 30p = 0$$

$$30p = \sin^{-1}(0) = n\pi$$

$$\boxed{p = \frac{n\pi}{30}}$$

Sub. $p = \frac{n\pi}{30}$ in (2)

$$u = BC \sin\left(\frac{n\pi x}{30}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{900}}$$

put $BC = b_n$

$$u = b_n \sin\left(\frac{n\pi x}{30}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{900}}$$

The most general solution is

$$u = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{30}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{900}} \rightarrow (3)$$

**Step: 3** Applying condition (iii) in (3)

$u = 2x + 20$ when $t = 0$

$$2x + 20 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{30}\right) e^{0}$$

$$2x + 20 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{30}\right)$$

(R.H.S. represent H.R.S.S.)

**To find $b_n$:-**

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \quad (\because l = 30)$$

$$b_n = \frac{2}{30} \int_0^{30} (2x+20) \sin\left(\frac{n\pi x}{30}\right) dx$$

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$$b_n = \frac{2}{15}\int_0^{30} (x+10)\sin\left(\frac{n\pi x}{30}\right)dx$$

By using Bernoulli's theorem,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots$$

$u = x+10$ $\quad v = \sin\left(\frac{n\pi x}{30}\right)$

$u' = 1$ $\quad\quad v_2 = \frac{-\sin\left(\frac{n\pi x}{30}\right)}{\left(\frac{n\pi}{30}\right)^2}$

$u'' = 0$ $\quad v_1 = \frac{-\cos\left(\frac{n\pi x}{30}\right)}{\frac{n\pi}{30}}$

$$b_n = \frac{2}{15}\left[\frac{-(x+10)\cos\left(\frac{n\pi x}{30}\right)}{\left(\frac{n\pi}{30}\right)} + \frac{\sin\left(\frac{n\pi x}{30}\right)}{\left(\frac{n\pi}{30}\right)^2}\right]_0^{30}$$

$$b_n = \frac{2}{15}\left[\frac{-40\cos n\pi}{\frac{n\pi}{30}} + 0 + \frac{10\cos 0}{\frac{n\pi}{30}} + 0\right]$$

$$b_n = \frac{2}{15}\left[\frac{-30\times 40}{n\pi}\cos n\pi + \frac{30\times 10}{n\pi}\cos 0\right]$$

$$b_n = \frac{2\times \overset{2}{30}\times 10}{15\, n\pi}\left[1-4(-1)^n\right]$$

$$b_n = \frac{40}{n\pi}\left(1-4(-1)^n\right)$$

Sub $b_n$ in (3)

$$u = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left(1-4(-1)^n\right) e^{-\frac{\alpha^2 n^2\pi^2 t}{900}} \sin\left(\frac{n\pi x}{30}\right)$$

$$u = \frac{40}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(1-4(-1)^n\right) e^{-\frac{\alpha^2 n^2\pi^2 t}{900}} \sin\left(\frac{n\pi x}{30}\right).$$

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# Type : 2 Temperature at both ends zero

Problem: 14

Find the solution to the eqn. $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ that satisfies the conditions $u(0,t) = 0$, $u(l,t) = 0$ $\forall$ $t > 0$ and

$$u(x,0) = \begin{cases} x, & 0 \le x \le l/2 \\ l-x, & l/2 \le x \le l \end{cases}$$

* Solution:

The one dimensional heat flow eqn. is

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

The solution is

$$u = (A\cos px + B\sin px)\, c e^{-\alpha^2 p^2 t} \longrightarrow (1)$$

The conditions are

i) $u = 0$ when $x = 0$

ii) $u = 0$ when $x = l$

iii) $u = f(x) = \begin{cases} x \ ; & 0 \le x \le l/2 \\ l-x \ ; & l/2 \le x \le l \end{cases}$ when $t = 0$

Step: 1 Applying cond. (i) in (1)

$u = 0$ when $x = 0$

$$0 = (A\cos 0 + B\sin 0)\, c e^{-\alpha^2 p^2 t}$$

$$0 = A c e^{-\alpha^2 p^2 t}$$

If $c \ne 0$, $e^{-\alpha^2 p^2 t} \ne 0$ $\therefore$ $A = 0$

Sub. $A = 0$ in (1)

$$u = B\sin px \cdot c e^{-\alpha^2 p^2 t} \longrightarrow (2)$$

Step: 2 Applying condition (ii) in (2)

$u = 0$ when $x = l$

$$0 = B\sin pl\, c e^{-\alpha^2 p^2 t}$$

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If $BC \neq 0 \quad e^{-\alpha^2 p^2 t} \neq 0$

$\therefore \sin pl = 0$

$pl = \sin^{-1}(0) = n\pi$

$p = \frac{n\pi}{l}$

Sub. $p = \frac{n\pi}{l}$ in (2)

$$u = BCe^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \cdot \sin\left(\frac{n\pi x}{l}\right)$$

Put $BC = b_n$

$$u = b_n e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}} \sin\left(\frac{n\pi x}{l}\right)$$

The most general solution is

$$u = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{l^2} t} \longrightarrow (3)$$

Step: 3 Applying cond. (iii) in (3)

$u = f(x)$ when $t = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-0}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Here R.H.S represents H.R.S.S.

To find $b_n$:

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{2}{l} \left\{ \int_0^{l/2} x \sin\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^{l} (l-x) \sin\left(\frac{n\pi x}{l}\right) dx \right\}$$

By using Bernoulli's theorem,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

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$u = x \quad u = l - x \quad v = \sin\left(\frac{n\pi x}{l}\right)$

$u' = 1 \quad u' = -1 \quad v_1 = \frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}$

$u'' = 0 \quad u'' = 0$

$v_2 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$

$$b_n = \frac{2}{l}\left\{\left[\frac{-x\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} + \frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_0^{l/2} + \left[\frac{-(l-x)\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - \frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_{l/2}^{l}\right\}$$

$$b_n = \frac{2}{l}\left\{\cancel{\frac{-l/2\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2} + \cancel{\frac{l/2\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2}\right\}$$

$$b_n = \frac{2}{\cancel{l}} \times \frac{l^{\cancel{2}}}{n^2\pi^2} \times 2 \sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{4l}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \begin{cases} \frac{4l}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) & \text{if } n = \text{odd} \\ 0, & \text{if } n = \text{even} \end{cases}$$

Sub $b_n$ in (3)

$$u = \sum_{n=odd}^{\infty} \frac{4l}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \cdot e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}} \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore u = \frac{4l}{\pi^2}\sum_{n=odd}^{\infty} \frac{e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}}{n^2}\sin\left(\frac{n\pi}{2}\right) \cdot \sin\left(\frac{n\pi x}{l}\right)$$

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Type : 3

Temperature at Both ends Non-zero and steady state Condition.

Problem :- 15

The ends A and B of a rod 30 cms long have their temperatures kept at 20°c & 80°c respectively until steady state Conditions prevail. The temperature at the end B is suddenly reduced to 60°c and of the A is raised to 40°c and maintained so.

Find $u(x,t)$.

* Solution :-

One Dimensional Heat Flow Equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

when steady state Condition prevails

$$\frac{\partial u}{\partial t} = 0$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = 0$$

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Step : 1

To find temperature

$u(x) = ax + b \rightarrow (A)$


At $x = 0$ when $u = 20$ sub. in (A)

$20 = a(0) + b$

$\boxed{b = 20}$

At $x = 30$ when $u = 80$ sub. in (A)

$80 = a(30) + 20$

$30a = 80 - 20$

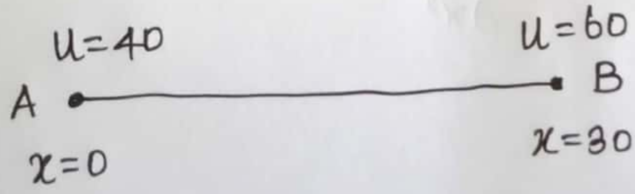
$30a = 60$

$\boxed{a = 2}$

Sub. a & b in (1)

$\boxed{u = 2x + 20}$

Step : 2

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We have the following Conditions

(i) $u = 40$ when $x = 0$

(ii) $u = 60$ when $x = l = 30$

(iii) $u = 2x + 20$ when $t = 0$

Here 1st and 2nd Conditions are non-zero. Hence we cannot solve the ~~first~~ heat flow equation.

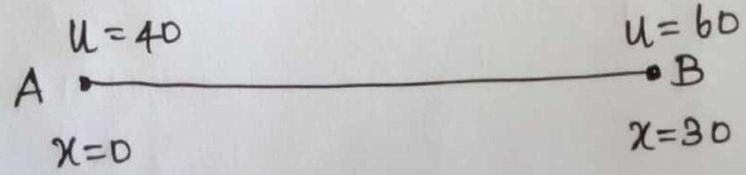
<u>step : 3</u>

We split the solution into 2 parts are

$$\underbrace{u(x,t) = u_t(x,t)}_{\text{Unsteady state}} + \underbrace{u_s(x)}_{\text{steady state}} \rightarrow (B)$$

<u>step : 4</u> To find $u_s(x)$ we have to apply the definition of steady state Conditions

$$u_s(x) = ax + b \longrightarrow (C)$$

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At $x=0$ when $u=40$ sub. in ~~(B)~~ (C)

$$40 = a(0) + b$$

$$\boxed{b = 40}$$

At $x=30$ when $u=60$ sub. in ~~(B)~~ (C)

$$60 = a(30) + 40$$

$$30a = 60 - 40$$

$$30a = 20$$

$$\boxed{a = 2/3}$$

Sub. a & b in ~~(B)~~ (C)

$$\boxed{u_s(x) = \frac{2}{3}x + 40}$$

<u>Step: 5</u>

To find $u_t(x,t)$ using (B)

we get

$$\boxed{u_t(x,t) = u(x,t) - u_s(x)} \rightarrow (D)$$

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Step: 6

We can solve for $u_t(x,t)$

at $x=0$, $x=30$ & $t=0$

At $x=0$

$$u_t(x,t) = u(x,t) - u_s(x)$$

$$u_t(0,t) = u(0,t) - u_s(0)$$

$$= 40 - 40$$

$$\boxed{u_t(0,t) = 0}$$

At $x=30$

$$u_t(x,t) = u(x,t) - u_s(x)$$

$$u_t(30,t) = u(30,t) - u_s(30)$$

$$= 60 - 60$$

$$\boxed{u_t(30,t) = 0}$$

At $t=0$

$$u_t(x,t) = u(x,t) - u_s(x)$$

$$u_t(x,0) = u(x,0) - u_s(x)$$

$$= 2x+20 - \left(\frac{2}{3}x+40\right)$$

$$= 2x+20 - \frac{2}{3}x - 40$$

$$\boxed{u_t(x,0) = \frac{4x}{3} - 20}$$

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Now we have the following boundary Conditions to solve for $u_t$.

(i) $u_t = 0$ when $x = 0$

(ii) $u_t = 0$ when $x = 30$

(iii) $u_t = \frac{4x}{3} - 20$ when $t = 0$

Step : 7

$$u_t = (A \cos px + B \sin px) c e^{-\alpha^2 p^2 t} \rightarrow (1)$$

Applying Condition (i) in (1)

$u_t = 0$ & $x = 0$

$$0 = (A \cos 0 + B \sin 0) c e^{-\alpha^2 p^2 t}$$

$$0 = A c e^{-\alpha^2 p^2 t}$$

$\therefore$ Here $c \neq 0$ & $e^{-\alpha^2 p^2 t} \neq 0$

$\therefore \boxed{A = 0}$

Sub. $A = 0$ in (1)

$$u_t = (0 + B \sin px) c e^{-\alpha^2 p^2 t}$$

$$u_t = B \sin px \cdot c e^{-\alpha^2 p^2 t} \longrightarrow (2)$$

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Step: 8

Applying Condition (ii) in (2)

$u_t = 0$ & $x = 30$

$$0 = B \sin p30 \cdot c e^{-\alpha^2 p^2 t}$$

$\therefore$ $B \neq 0$, $c \neq 0$ & $e^{-\alpha^2 p^2 t} \neq 0$

$$\therefore \quad \sin 30p = 0$$

$$30p = \sin^{-1}(0)$$

$$30p = n\pi$$

$$\boxed{p = \frac{n\pi}{30}}$$

Sub. $p = \frac{n\pi}{30}$ in (2)

$$u_t = BC \sin\left(\frac{n\pi x}{30}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{30^2} t}$$

put $BC = b_n$

$$u_t = b_n \sin\left(\frac{n\pi x}{30}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{30^2} t}$$

The most general solution is

$$u_t = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{30}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{30^2}} \quad \to (3)$$

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Step: 9

Applying Condition (iii) in (3)

$u_t = \frac{4x}{3} - 20$ & $t=0$

$$\frac{4x}{3} - 20 = \sum_{n=1}^{\infty} bn \sin\left(\frac{n\pi x}{30}\right) e^{0}$$

$$\frac{4x}{3} - 20 = \sum_{n=1}^{\infty} bn \sin\left(\frac{n\pi x}{30}\right)$$

(R.H.S. represent H.R.S.S)

To find bn :-

$$bn = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$l = 30$

$$= \frac{2}{30}\int_0^{30} \underbrace{\left(\frac{4x}{3} - 20\right)}_{u} \underbrace{\sin\left(\frac{n\pi x}{30}\right)}_{v} dx$$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

| | |
|---|---|
| $u = \frac{4x}{3} - 20$ | $v = \sin(n\pi x/30)$ |
| $u' = 4/3$ | $v_1 = \int v = \frac{-\cos(n\pi x/30)}{(n\pi/30)}$ |
| $u'' = 0$ | $v_2 = \int v_1 = \frac{-\sin(n\pi x/30)}{(n\pi/30)^2}$ |

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$$b_n = \frac{1}{15}\left[\left(\frac{4x}{3}-20\right)\left(\frac{-\cos\left(\frac{n\pi x}{30}\right)}{(n\pi/30)}\right)-\left(\frac{4}{3}\right)\left(\cancel{\frac{-\sin\left(\frac{n\pi x}{30}\right)}{(n\pi/30)^2}}\right)\right]_0^{30}$$

$$\because \sin n\pi = \sin 0 = 0$$

$$= \frac{1}{15}\left[-\frac{30}{n\pi}\left(\frac{4x}{3}-20\right)\cos\left(\frac{n\pi x}{30}\right)\right]_0^{30}$$

$$= \frac{1}{15}\left(\frac{-30}{n\pi}\right)\left[\left(\frac{4x}{3}-20\right)\cos\left(\frac{n\pi x}{30}\right)\right]_0^{30}$$

$$= \frac{-2}{n\pi}\left[(40-20)\cos n\pi - (-20)\cos 0\right]$$

$$= \frac{-2}{n\pi}\left[20(-1)^n + 20\right]$$

$$= \frac{-40}{n\pi}\left[(-1)^n + 1\right]$$

$$b_n = \begin{cases} \dfrac{-80}{n\pi}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

~~The Final Soluti~~

$$\therefore\ u_t = \sum_{n=\text{even}}^{\infty}\left(\frac{-80}{n\pi}\right)\sin\left(\frac{n\pi x}{30}\right)e^{-\frac{\alpha^2 n^2 \pi^2 t}{30^2}}$$

The Final Solution is

$$u(x,t) = u_t(x,t) + u_s(x)$$

$$u(x,t) = \frac{2}{3}x + 40 + \sum_{n=\text{even}}^{\infty}\left(\frac{-80}{n\pi}\right)\sin\left(\frac{n\pi x}{30}\right)e^{-\frac{\alpha^2 n^2 \pi^2 t}{30^2}}$$

— x —

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## * Method of Separation of Variables :-

**Problem :**

Solve the equation $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ given that $u(x,0) = 4e^{-x}$ by the method of Separation of Variables.

## * Solution :-

Given : $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 \rightarrow (1)$

$$u(x,0) = 4e^{-x}$$

when $y = 0$ ; $u = 4e^{-x}$

Consider

$$u = X(x) \cdot Y(y) \longrightarrow (2)$$

Diff. (2) partially w.r. to 'x & y'

$$\left.\begin{array}{l|l} \frac{\partial u}{\partial x} = X'Y & \frac{\partial u}{\partial y} = XY' \end{array}\right\} \rightarrow (3)$$

Sub. (3) in (1)

$$3X'Y + 2XY' = 0$$

$$3X'Y = -2XY'$$

Separating the Variables, we get

$$\frac{3X'}{X} = -\frac{2Y'}{Y}$$

$$\frac{3X'}{X} = \frac{-2Y'}{Y} = k \text{ (say)}$$

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Consider

$$\frac{3x'}{x} = k$$

$$3x' = kx$$

$$x' = \frac{kx}{3}$$

$$\frac{d}{dx}(x) = \frac{kx}{3}$$

$$\frac{dx}{x} = \frac{k}{3}\,dx$$

Integrating

$$\int \frac{dx}{x} = \frac{k}{3}\int dx$$

$$\log x = \frac{k}{3}x + \log a$$

$$\log x - \log a = \frac{k}{3}x$$

$$\log\left(\frac{x}{a}\right) = \frac{k}{3}x$$

$$\frac{x}{a} = e^{\frac{kx}{3}}$$

$$x = a\,e^{\frac{kx}{3}} \rightarrow (4)$$

Consider

$$\frac{-2y'}{y} = k$$

$$-2y' = yk$$

$$2y' = -yk$$

$$y' = -\frac{yk}{2}$$

$$\frac{d(y)}{dy} = -\frac{yk}{2}$$

$$\frac{dy}{y} = -\frac{k}{2}\,dy$$

Integrating

$$\int \frac{dy}{y} = -\frac{k}{2}\int dy$$

$$\log y = -\frac{k}{2}y + \log b$$

$$\log y - \log b = -\frac{k}{2}y$$

$$\log\left(\frac{y}{b}\right) = -\frac{k}{2}y$$

$$\frac{y}{b} = e^{\frac{-ky}{2}}$$

$$y = b\,e^{-ky/2} \longrightarrow (5)$$

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Sub. (4) & (5) in (1)

$$u = X \cdot Y$$

$$= a e^{kx/3} \cdot b e^{-ky/2}$$

$$u = ab\, e^{kx/3} \cdot e^{-ky/2} \rightarrow (6)$$

Given: $u = 4e^{-x}$ ; when $y = 0$

From (6)

$$4e^{-x} = ab\, e^{kx/3}\, e^{0}$$

$$4e^{-x} = ab\, e^{kx/3}$$

Comparing the Co-efficients on both side '$\underline{ab}$ & $\underline{x}$'

$$\left.\begin{aligned} \therefore \quad ab &= 4 \\ k/3 &= -1 \\ k &= -3 \end{aligned}\right\} \rightarrow (7)$$

Sub. (7) in (6)

$$u = 4e^{-x} \cdot e^{3y/2}$$

—— x ——

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* Derive one Dimensional wave equation by Separation of variables method.

One Dimensional wave eqn. is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \rightarrow (1)$$

let $y(x,t) = X(x)\,T(t) \rightarrow (2)$

be the solution of the given equation

$X \rightarrow$ function of 'x' only

$T \rightarrow$ function of 't' only

From (2)

$$\left.\begin{array}{l|l} \dfrac{\partial y}{\partial x} = X'T & \dfrac{\partial y}{\partial t} = XT' \\ \dfrac{\partial^2 y}{\partial x^2} = X''T & \dfrac{\partial^2 y}{\partial t^2} = XT'' \end{array}\right\} \rightarrow (3)$$

Sub. (3) in (1)

$$XT'' = a^2 X''T$$

$$\frac{T''}{a^2 T} = \frac{X''}{X} = K$$

$$\begin{array}{l|l} \dfrac{X''}{X} = K & \dfrac{T''}{a^2 T} = K \\ X'' = XK & T'' = a^2 KT \\ X'' - XK = 0 \rightarrow (4) & T'' - a^2 KT = 0 \rightarrow (5) \end{array}$$

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From (4) & (5), we get the solutions of K.
There are three cases arises.

<u>Case (i)</u> Let $K = p^2$ in (4) & (5)

$X'' - p^2 X = 0$

$\frac{d^2X}{dx^2} - p^2 X = 0$

The Auxillary eqn. is

$m^2 - p^2 = 0$

$m = \pm p$

$X = c_1 e^{px} + c_2 e^{-px}$ → (6)

$T'' - p^2a^2T = 0$

$\frac{d^2T}{dt^2} - p^2a^2T = 0$

The Auxillary eqn. is

$m^2 - p^2a^2 = 0$

$m = \pm ap$

$T = c_3 e^{pat} + c_4 e^{-pat}$ → (7)

Sub. (6) & (7) in (2)

$y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pat} + c_4 e^{-pat})$

<u>Case (ii)</u> put $K = -p^2$ in (4) & (5)

$X'' + p^2 X = 0$

$\frac{d^2X}{dx^2} + p^2 X = 0$

The Auxillary eqn. is

$m^2 + p^2 = 0$

$m = \pm ip$

$X = c_5 \cos px + c_6 \sin px$ → (8)

$T'' + p^2a^2T = 0$

$\frac{d^2T}{dt^2} + p^2a^2T = 0$

The Auxillary eqn. is

$m^2 + a^2p^2 = 0$

$m = \pm iap$

$T = c_7 \cos pat + c_8 \sin pat$ → (9)

Sub. (8) & (9) in (2)

$y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pat + c_8 \sin pat)$

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Case (iii)

put $k=0$ in (4) & (5)

$$X'' = 0$$

$$\frac{d^2X}{dx^2} = 0$$

Integrating twice w.r.to 'x'

$$X = C_9 x + C_{10} \rightarrow (10)$$

$$T'' = 0$$

$$\frac{d^2T}{dt^2} = 0$$

Integrating twice w.r.to 't'

$$T = C_{11} t + C_{12} \rightarrow (11)$$

Sub. (10) & (11) in (2)

$$y(x,t) = (C_9 x + C_{10})(C_{11} t + C_{12})$$

Thus depending upon the value of k, Various possible solutions of the wave equations are

$$y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pat} + c_4 e^{-pat})$$

$$y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pat + c_8 \sin pat)$$

$$y(x,t) = (C_9 x + C_{10})(C_{11} t + C_{12})$$

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* Derive one Dimensional Heat Flow equation by Separation of Variable method.

One Dimensional Heat Flow eqn. is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$$

Let $u(x,t) = X(x)\, T(t) \rightarrow (2)$

From (2)

$$\left.\begin{array}{l|l} \dfrac{\partial u}{\partial x} = X'T & \dfrac{\partial u}{\partial t} = XT' \\ \dfrac{\partial^2 u}{\partial x^2} = X''T & \end{array}\right\} \rightarrow (3)$$

Sub. (3) in (1)

$$XT' = a^2 X''T$$

$$\frac{T'}{a^2 T} = \frac{X''}{X} = K \text{ (Say)}$$

| | |
|---|---|
| $\dfrac{T'}{a^2 T} = K$ | $\dfrac{X''}{X} = K$ |
| $T' = K a^2 T$ | $X'' = KX$ |
| $T' - K a^2 T = 0 \rightarrow (4)$ | $X'' - KX = 0 \rightarrow (5)$ |

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From (4) & (5), we get the solutions of K. There are three cases arises.

Case :(i)

put $K = p^2$ in (4) & (5)

$X'' - p^2 X = 0$

$\frac{d^2 X}{dx^2} - p^2 X = 0$

The Auxillary eqn. is

$m^2 - p^2 = 0$

$m^2 = p^2$

$m = \pm p$

$X = c_1 e^{px} + c_2 e^{-px} \rightarrow (6)$

$T' - p^2 \alpha^2 T = 0$

$\frac{dT}{dt} - p^2 \alpha^2 T = 0$

$\frac{dT}{dt} = p^2 \alpha^2 T$

$\frac{dT}{T} = p^2 \alpha^2 dt$

$\int \frac{dT}{T} = p^2 \int dt$

$\log T = \alpha^2 p^2 t + \log c_3$

$\log T - \log c_3 = \alpha^2 p^2 t$

$\log\left(\frac{T}{c_3}\right) = \alpha^2 p^2 t$

$\frac{T}{c_3} = e^{\alpha^2 p^2 t}$

$T = c_3 e^{\alpha^2 p^2 t} \rightarrow (7)$

Sub. (6) & (7) in (2)

$u(x,t) = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{\alpha^2 p^2 t}$

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Case (ii)

put $k = -p^2$ in (4) & (5)

$$X'' + p^2 X = 0$$

$$\frac{d^2X}{dx^2} + p^2 X = 0$$

The Auxillary eqn. is

$$m^2 + p^2 = 0.$$

$$m^2 = -p^2$$

$$m = \pm ip$$

$$X = c_4 \cos px + c_5 \sin px \quad \rightarrow (8)$$

$$T' + p^2\alpha^2 T = 0$$

$$\frac{dT}{dt} = -p^2\alpha^2 T$$

$$\frac{dT}{T} = -\alpha^2 p^2 \, dt$$

$$\int \frac{dT}{T} = -\alpha^2 p^2 \int dt$$

$$\log T = -\alpha^2 p^2 t + \log c_6$$

$$\log T - \log c_6 = -\alpha^2 p^2 t$$

$$\log\left(\frac{T}{c_6}\right) = -\alpha^2 p^2 t$$

$$\frac{T}{c_6} = e^{-\alpha^2 p^2 t}$$

$$T = c_6 e^{-\alpha^2 p^2 t} \quad \rightarrow (9)$$

Sub. (8) & (9) in (2)

$$u(x,t) = (c_4 \cos px + c_5 \sin px)\, c_6 e^{-\alpha^2 p^2 t}$$

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Case (iii) put $K=0$ in (4) & (5)

$X'' = 0$

$\frac{d^2X}{dx^2} = 0$

Integrating twice w.r. to 'x'

$X = C_7 x + C_8 \rightarrow (10)$

$T' = 0$

$\frac{dT}{dt} = 0$

$dT = 0$

$\int dT = 0$

$T = C_9 \rightarrow (11)$

Sub. (10) & (11) in (2)

$$u(x,t) = (C_7 x + C_8)\, C_9$$

Thus depend upon the value of K, Various possible solutions of the Heat flow equations are

$$u(x,t) = (C_1 e^{px} + C_2 e^{-px})\, C_3 e^{\alpha^2 p^2 t}$$

$$u(x,t) = (C_4 \cos px + C_5 \sin px)\, C_6 e^{-\alpha^2 p^2 t}$$

$$u(x,t) = (C_7 x + C_8)\, C_9$$

x

* Derive Two Dimensional Heat equation by separation of Variable method.

Two Dimensional Heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow (1)$$

Let $u = X(x)\, Y(y) \rightarrow (2)$

From (2)

$$\left.\begin{array}{l|l} \frac{\partial u}{\partial x} = X'Y & \frac{\partial u}{\partial y} = XY' \\ \frac{\partial^2 u}{\partial x^2} = X''Y & \frac{\partial^2 u}{\partial y^2} = XY'' \end{array}\right\} \rightarrow (3)$$

Sub. (3) in (1)

$$X''Y + XY'' = 0$$

$$X''Y = -XY''$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = k \text{ (Say)}$$

$$\begin{array}{l|l} \frac{X''}{X} = k & -\frac{Y''}{Y} = k \\ X'' = kX & -Y'' = kY \\ X'' - Xk = 0 \rightarrow (4) & Y'' + kY = 0 \rightarrow (5) \end{array}$$

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Case (i) put $k = p^2$ in (4) & (5)

$x'' - p^2 x = 0$

$\frac{d^2x}{dx^2} - p^2 x = 0$

The Auxillary eqn. is

$m^2 - p^2 = 0$

$m^2 = p^2$

$m = \pm p$

$X = c_1 e^{px} + c_2 e^{-px}$ → (6)

$y'' + p^2 y = 0$

$\frac{d^2y}{dy^2} + p^2 y = 0$

The Auxillary eqn. is

$m^2 + p^2 = 0$

$m^2 = -p^2$

$m = \pm ip$

$Y = c_3 \cos py + c_4 \sin py$ → (7)

Sub. (6) & (7) in (2)

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

Case (ii) put $k = -p^2$ in (4) & (5)

$x'' + p^2 x = 0$

$\frac{d^2x}{dx^2} + p^2 x = 0$

The Auxillary eqn. is

$m^2 + p^2 = 0$

$m^2 = -p^2$

$m = \pm ip$

$X = c_5 \cos px + c_6 \sin px$ → (8)

$y'' - p^2 y = 0$

$\frac{d^2y}{dy^2} - p^2 y = 0$

The Auxillary eqn. is

$m^2 - p^2 = 0$

$m^2 = p^2$

$m = \pm p$

$Y = c_7 e^{py} + c_8 e^{-py}$ → (9)

Sub (8) & (9) in (2)

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$$

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<u>Case (iii)</u> put $k=0$ in (4) & (5)

$X'' = 0$

$\frac{d^2X}{dx^2} = 0$

Integrating twice w.r. to 'x'

$X = c_9 x + c_{10} \longrightarrow (10)$

$Y'' = 0$

$\frac{d^2Y}{dy^2} = 0$

Integrating twice w.r.to 'y'

$Y = c_{11} y + c_{12} \rightarrow (11)$

Sub (10) & (11) in (2)

$$u = (c_9 x + c_{10})(c_{11} y + c_{12})$$

∴ The various possible solutions are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12})$$

X

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